Quantization of Field Theory on the Light Front

V. A. Franke, Yu. V. Novozhilov, S. A. Paston, E. V. Prokhvatilov
St.-Petersburg State University, Russia

Abstract

Canonical formulation of quantum field theory on the Light Front (LF) is reviewed. The problem of constructing the LF Hamiltonian which gives the theory equivalent to original Lorentz and gauge invariant one is considered. We describe possible ways of solving this problem: (a) the limiting transition from the equal-time Hamiltonian in a fast moving Lorentz frame to LF Hamiltonian, (b) the direct comparison of LF perturbation theory in coupling constant and usual Lorentz-covariant Feynman perturbation theory. The results of the application of method (b) to QED-1+1 and QCD-3+1 are given. Gauge invariant regularization of LF Hamiltonian via introducing a lattice in transverse coordinates and imposing periodic boundary conditions in LF coordinate x^- for gauge fields on the interval $|x^-| < L$ is also considered.

1 Introduction

A possible approach to solving the strong interaction field theory is the canonical quantization on the Light-Front (LF) with the application of corresponding Schroedinger equation. To realize such program one introduces LF coordinates [1] $x^{\pm} = (x^0 \pm x^3)/\sqrt{2}, x^1, x^2$, where x^0, x^1, x^2, x^3 are Lorentz coordinates. The x^+ plays the role of time, and canonical quantization is carried out on a hypersurface $x^+ = const$. The advantage of this scheme is connected with the positivity of the momentum P_- (translation operator along x^- axis), which becomes quadratic in fields on the LF. As a consequence the lowest eigenstate of the operator P_- is both physical vacuum and the "mathematical" vacuum of perturbation theory [2]. Using Fock space [3] over this vacuum one can solve stationary Schroedinger equation with Hamiltonian P_+ (translation operator along x^+ axis) to find the spectrum of bound states. The problem of describing the physical vacuum, very complicated in usual formulation with Lorentz coordinates, does not appear here. Such approach is called LF Hamiltonian approach. It attracts attention for a long time as a possible mean for solving Quantum Field Theory problems.

While giving essential advantages, the application of LF coordinates in Quantum Field Theory leads to some difficulties. The hyperplane $x^+ = \text{const}$ is a characteristic surface for relativistic differential field equations. It is not evident without additional investigation that quantization on such hypersurface generates a theory equivalent to one quantized in the usual way in Lorentz coordinates [4–12]. This is in particular essential because of the special divergences at $p_- = 0$ appearing in LF quantization scheme. Beside of usual ultraviolet regularization one has to apply special regularization of such divergences. We will consider the following simplest prescription of such regularization:

(a) cutoff of momenta p_-

$$|p_-| \geq \varepsilon, \quad \varepsilon > 0; \tag{1}$$

(b) cutoff of the x^-

$$-L \leq x^- \leq L. \tag{2}$$

with periodic boundary conditions in x^- for all fields.

The regularization (b) discretizes the spectrum of the operator P_- ($p_- = \pi n/L$, where n is an integer). This formulation is called sometime "Discretized LF Quantization" [13]. Fourier components of fields, corresponding to $p_- = 0$ (and usually called "zero modes") turn out to be dependent variables and must be expressed in terms of nonzero modes via solving constraint equations (constraints). These constraints are usually very complicated, and solving of them is a difficult problem.

The prescriptions of regularization of divergences at $p_- = 0$ described above are convenient for Hamiltonian approach, but both of them break Lorentz invariance and the prescription (a) breaks also the gauge invariance. Therefore the equivalence of LF and original Lorentz (and gauge) invariant formulation can be broken even in the limit of removed cutoff. To avoid this nonequivalence some modification of usual renormalization procedure may be necessary, see for example [11, 14] and [15]. The problem of constructing the LF Hamiltonian which gives a theory equivalent to original Lorentz and gauge invariant one turned out to be rather difficult. We will describe possible approaches to this problem.

In sect. 2 we give basic relations of quantum field theory in LF coordinates. In sect. 3 we consider the limiting transition from fast moving Lorentz frame to the LF. This transition relates Hamiltonian formulations in Lorentz and LF coordinates and firstly was presented in [7, 8]. In sect. 4 we investigate the relation between LF perturbation theory in coupling constant and usual Lorentz-covariant Feynman perturbation theory. With the help of this investigation we show how to construct LF Hamiltonian giving a theory perturbatively equivalent to original one for Yukawa model, for QCD in four-dimensional space-time, and for QED in two-dimensional space-time (originally this was considered in [11, 14, 16–18]. In sect. 5 we carry out gauge invariant ultraviolet regularization of LF Hamiltonian introducing a lattice in transverse coordinates x^1, x^2 and using, instead of transverse components of gauge fields, complex matrices in color space as independent gauge variables on the links of the lattice [19, 20]. We find that LF canonical formalism for gauge theories with this regularization avoid usual most complicated constraints connecting zero and nonzero modes. However, the quantization leads to Lorentz-noninvariant results.

2 Formal canonical quantization of Field Theory on the Light Front and the problem of bound states

In order to find the bound state spectrum in some field theory quantized on the LF the following system of equations is usually solved:

$$P_+|\Psi\rangle = P'_+|\Psi\rangle, \tag{3}$$

$$P_-|\Psi\rangle = P'_-|\Psi\rangle, \tag{4}$$

$$P_\perp|\Psi\rangle = 0, \tag{5}$$

where $P_\perp = \{P_1, P_2\}$. The mass of bound state is equal to

$$m = \sqrt{2P'_+P'_-}. \tag{6}$$

It was taken into account that nonzero components of metric tensor in LF coordinates are

$$g_{-+} = g_{+-} = 1, \quad g_{11} = g_{22} = -1. \tag{7}$$

The operators P_-, $P_\perp$ are quadratic in fields, and the solution of equations (4), (5) is trivial. The problem is in solving the Schroedinger equation (3). Physical vacuum $|\Omega\rangle$ is lowest eigenstate of the operator P_-, and must satisfy the equations:

$$P_+|\Omega\rangle = 0, \tag{8}$$

$$P_-|\Omega\rangle = 0, \tag{9}$$

$$P_\perp|\Omega\rangle = 0. \tag{10}$$

To fulfil equations (8), (9) one should subtract , if it is necessary, corresponding renormalizing constants from the operators P_+, P_-. The $|\Omega\rangle$ plays simultaneously the role of mathematical vacuum of LF Fock space. A solution $|\Psi\rangle$ of Schrodinger equation (3) belongs to this space.

Expressions for the operators P_+, P_- can be obtained by canonical quantization on the LF. Let us describe the procedure of such quantization via some examples, without analyzing so far the question about the equivalence of appearing theory and original Lorentz-covariant one. It is assumed that in addition to explicit regularization of divergences at $p_- = 0$ also ultraviolet regularization is implied.

2.1 Scalar selfinteracting field in (1+1)-dimensional space-time.

Peculiarities of LF quantization are well seen even in this simple example. We have only LF coordinates x^+, x^-. The Lagrangian is equal to

$$L = \int dx^- \left(\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \lambda \varphi^4 \right), \tag{11}$$

or

$$L = \int dx^- \left(\frac{1}{2} \partial_+ \varphi \partial_- \varphi - \frac{1}{2} m^2 \varphi^2 - \lambda \varphi^4 \right), \tag{12}$$

The "time" derivative $\partial_+ \varphi$ enters into this Lagrangian only linearly. For the transition to canonical theory it is sufficient to rewrite the expression $\int dx^- \frac{1}{2} \partial_+ \varphi \partial_- \varphi$ in standard canonical form. To achieve this let us take the Fourier decomposition

$$\varphi(x^-) = (2\pi)^{-\frac{1}{2}} \int_0^\infty dk \, |2k|^{-\frac{1}{2}} \left(a(k) \exp(-ikx^-) + a^+(k) \exp(ikx^-) \right), \tag{13}$$

where $k \equiv k_-$, $\varphi(x^-) \equiv \varphi(x^+, x^-)$, $a(k) \equiv a(x^+, k)$. The Lagrangian (12) takes the form

$$L = \int_0^\infty dk \left(\frac{a^+(k) \dot{a}(k) - a(k) \dot{a}^+(k)}{2i} \right) - H, \tag{14}$$

where $\dot{a} \equiv \partial a / \partial x^+$ and

$$H = \int dx^- \left(\frac{1}{2} m^2 \varphi^2 + \lambda \varphi^4 \right). \tag{15}$$

Here we have used the equality

$$\int_0^\infty dk \int_0^\infty dk' \, \delta(k + k') k' \left(a(k) \dot{a}(k') - a^+(k) \dot{a}^+(k') \right) = 0. \tag{16}$$

It is implied that the function $\varphi(x)$ in (15) is expressed in terms of $a^+(k)$ and $a(k)$ with the help of formulae (12). "Time" derivatives $\dot{a}(k)$, $\dot{a}^+(k)$ enter into Lagrangian L in a form standard for canonical theory. Therefore one can interpret after quantization the $a^+(k)$ and $a(k)$ as creation and annihilation operators satisfying the following commutation relations at fixed x^+ and $k > 0$, $k' > 0$:

$$[a(k), a^+(k')] = \delta(k - k'), \quad [a(k), a(k')] = 0, \tag{17}$$

It is also seen that the H is LF Hamiltonian, i.e. $H = P_+$.

We have also the formulae

$$P_\mu = \int dx^- \, T_{-\mu}, \tag{18}$$

where the energy-momentum tensor $T_{\nu\mu}$ is equal to

$$T_{\nu\mu} = \partial_\nu \varphi \partial_\mu \varphi - g_{\mu\nu} \mathcal{L}. \tag{19}$$

Via this relation one can reproduce the expression (15) for $P_+ \equiv H$, and obtain the equality

$$P_- = \int dx^- \, (\partial_- \varphi)^2 = \frac{1}{2} \int\limits_0^\infty dk \, k \left(a^+(k) a(k) + a(k) a^+(k) \right) . \tag{20}$$

The lowest eigenstate of the operator P_- is the physical vacuum $|\Omega\rangle$ for which

$$a(k)|\Omega\rangle = 0 \tag{21}$$

at any k. It is seen that vacuum expectation values $\langle\Omega|P_-|\Omega\rangle$, $\langle\Omega|P_+|\Omega\rangle$ are infinite. The renormalization can be got by taking normal ordered forms $:P_+:$, $:P_-:$ with respect to the operators a^+, a (the symbol :: means as usual that operators a^+ stand everywhere before of operators a). Normal ordering of the $\lambda\varphi^4$ term in the Hamiltonian P_+ leads also to renormalization of the mass. In the following, writing P_+, P_-, we mean the expressions $:P_+:$, $:P_-:$, satisfying conditions (8), (9). Normal ordering of the operators P_+, P_- allows to avoid all ultraviolet divergences in this simple model.

2.2 Theory of interacting scalar and fermion fields in (3+1)-dimensional space-time (Yukawa model)

The Lagrangian of the model is

$$L = \int d^2x^\perp dx^- \left(\overline{\psi} \left(i\gamma^\mu \partial_\mu - M \right) \psi + \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - g\overline{\psi}\psi\varphi - \lambda' \varphi^3 - \lambda \varphi^4 \right), \tag{22}$$

where M is the fermion mass, m is the boson mass, $\overline{\psi} = \psi^+ \gamma^0$, $\varphi = \varphi^+$; g, λ, λ' are coupling constants. Here and so on $\mu, \nu, \ldots = +, -, 1, 2$; $i, k, \ldots = 1, 2$; $x^\perp \equiv (x^1, x^2)$. For Dirac's γ-matrices we use:

$$\gamma^0 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & iI \\ iI & 0 \end{pmatrix}, \qquad \gamma^\perp = \begin{pmatrix} -i\sigma^\perp & 0 \\ 0 & i\sigma^\perp \end{pmatrix}, \tag{23}$$

where I is a unit 2×2 matrix, $\sigma^\perp \equiv \{\sigma^1, \sigma^2\}$, σ^i are Pauli matrices.

We introduce 2-component spinors χ, ξ, writing

$$\psi = \begin{pmatrix} \chi \\ \xi \end{pmatrix}. \tag{24}$$

The Lagrangian L can be written in the form

$$\begin{aligned} L = \int d^2x^\perp dx^- \Big(& i\sqrt{2}\chi^+ \partial_+ \chi + i\sqrt{2}\xi^+ \partial_- \xi + \left(i\xi^+ \left(\sigma^i \partial_i - M \right) \chi + \text{h.c.} \right) + \\ & + \partial_+ \varphi \partial_- \varphi - \frac{1}{2} \partial_i \varphi \partial_i \varphi - \frac{1}{2} m^2 \varphi^2 - ig\varphi \left(\xi^+ \chi - \chi^+ \xi \right) - \lambda' \varphi^3 - \lambda \varphi^4 \Big), \end{aligned} \tag{25}$$

where h.c. means Hermitian conjugation. The variation of this Lagrangian with respect to χ^+ leads to the equation

$$\sqrt{2} \partial_- \xi = -\left(\sigma^i \partial_i - M \right) \chi + g\varphi\chi. \tag{26}$$

This equation does not contain derivatives in x^+ and therefore is a constraint. One should solve it with respect to ξ and substitute the result into the Lagrangian. In doing this we must invert the operator ∂_- which becomes an operator of multiplication ik_- after Fourier transformation. Inverse operator $(ik_-)^{-1}$ has singularity at $k_- = 0$. To avoid this singularity we introduce the regularization (1). For any function $f(x^-) \equiv f(x^+, x^-, x^\perp)$ we define Fourier transform

$$f(x^-) = \frac{1}{\sqrt{2\pi}} \int dk_- \, e^{ik_- x^-} \tilde{f}(k_-), \tag{27}$$

where $\tilde{f}(k_-) \equiv \tilde{f}(x^+, k_-, x^\perp)$, and put

$$[f(x^-)] \equiv \frac{1}{\sqrt{2\pi}} \int dk_- \, e^{ik_- x^-} \tilde{f}(k_-), \qquad |k_-| \geq \varepsilon > 0. \tag{28}$$

We insert into the Lagrangian (25) the variables $[\chi]$, $[\chi^+]$, $[\xi]$, $[\xi^+]$, $[\varphi]$ instead of χ, χ^+, ξ, ξ^+, φ and obtain the constraint equation

$$\sqrt{2}\partial_-[\xi] = -\left(\sigma^i\partial_i - M\right)[\chi] + g[[\varphi][\chi]] \tag{29}$$

instead of (26). Its solution is

$$[\xi] = \frac{1}{\sqrt{2}}\partial_-^{-1}\Big(-\left(\sigma^i\partial_i - M\right)[\chi] + g[[\varphi][\chi]]\Big), \tag{30}$$

where the operator ∂_-^{-1} is completely defined by the condition

$$\partial_-^{-1}[f] = \left[\partial_-^{-1}[f]\right]. \tag{31}$$

After Fourier transformation the operator ∂_-^{-1} is replaced by $(ik_-)^{-1}$.

Substituting the expression (30) into the Lagrangian (where all fields χ, $\chi^+, \ldots$ are replaced with $[\chi]$, $[\chi^+], \ldots$) we come to the result:

$$\begin{aligned} L = \int d^2x^\perp dx^- \Big(& i\sqrt{2}\left[\chi^+\right]\partial_+[\chi] + \partial_-[\varphi]\partial_+[\varphi] + \\ & + \frac{1}{\sqrt{2}}\Big(\left(\sigma^i\partial_i - M\right)[\chi] - g[[\varphi][\chi]]\Big)^+ (-i\partial_-)^{-1}\Big(\left(\sigma^k\partial_k - M\right)[\chi] - g[[\varphi][\chi]]\Big) - \\ & - \frac{1}{2}\partial_i[\varphi]\partial_i[\varphi] - \frac{1}{2}m^2[\varphi]^2 - \lambda'[\varphi]^3 - \lambda[\varphi]^4\Big), \end{aligned} \tag{32}$$

As in sect. 2a time derivatives $\partial_+[\chi]$, $\partial_+[\varphi]$ enter into the Lagrangian (32) linearly. Therefore to go to canonical formalism it is sufficient to find a standard canonical form for the expression

$$i\sqrt{2}[\chi^+]\partial_+[\chi] + \partial_-[\varphi^+]\partial_+[\varphi] \tag{33}$$

(before a quantization the quantities χ^+, χ are elements of Grassman algebra). We write

$$[\varphi(x^-)] = (2\pi)^{-1/2} \int_\varepsilon^\infty dk_- \, (2k_-)^{-1/2} (a(k_-)\exp(-ik_-x^-) + a^+(k_-)\exp(ik_-x^-)), \tag{34}$$

$$[\chi_r(x^-)] = (2\pi)^{-1/2} 2^{-1/4} \int_\varepsilon^\infty dk_- (b_r(k_-)\exp(-ik_-x^-) + c_r^+(k_-)\exp(ik_-x^-)), \tag{35}$$

$$[\chi_r^+(x^-)] = (2\pi)^{-1/2}2^{-1/4}\int\limits_{\varepsilon}^{\infty} dk_-(c_r(k_-)\exp(-ik_-x^-) + b_r^+(k_-)\exp(ik_-x^-)), \tag{36}$$

where

$$[\varphi(x^-)] \equiv [\varphi(x^+, x^-, x^\perp)], \qquad a(k_-) \equiv a(x^+, k_-, x^\perp) \tag{37}$$

et cetera, $r = 1, 2$. The Lagrangian (32) takes the form

$$L = \int d^2x^\perp \int\limits_{\varepsilon}^{\infty} dk_-\,(a(k_-)\dot{a}^+(k_-) - a^+(k_-)\dot{a}(k_-) - \\ -b_r(k_-)\dot{b}_r^+(k_-) - b_r^+(k_-)\dot{b}_r(k_-) - c_r(k_-)\dot{c}_r^+(k_-) - c_r^+(k_-)\dot{c}_r(k_-)) - H, \tag{38}$$

where

$$H = \Big(-\frac{1}{\sqrt{2}}\Big(\Big(\sigma^i\partial_i - M\Big)[\chi] - g[[\varphi][\chi]]\Big)^+ (-i\partial_-)^{-1}\Big(\Big(\sigma^k\partial_k - M\Big)[\chi] - g[[\varphi][\chi]]\Big) + \\ +\frac{1}{2}\partial_i[\varphi]\partial_i[\varphi] + \frac{1}{2}m^2[\varphi]^2 + \lambda'[\varphi]^3 + \lambda[\varphi]^4\Big). \tag{39}$$

It is assumed that the quantities $[\chi]$, $[\chi^+]$, $[\varphi]$ in the formulae (39) are expressed in terms of b, b^+, c, c^+, a, a^+ with the help of (34), (35), (36).

It follows from (38) that a^+, a, b^+, b, c^+, c play a role of creation and annihilation operators. After quantization they satisfy the commutation relations (at x^+ =const):

$$[a(k_-, x^\perp), a^+(k'_-, x'^\perp)]_- = \delta(k_- - k'_-)\delta^2(x^\perp - x'^\perp), \tag{40}$$

$$[b(k_-, x^\perp), b^+(k'_-, x'^\perp)]_+ = \delta(k_- - k'_-)\delta^2(x^\perp - x'^\perp), \tag{41}$$

$$[c(k_-, x^\perp), c^+(k'_-, x'^\perp)]_+ = \delta(k_- - k'_-)\delta^2(x^\perp - x'^\perp), \tag{42}$$

where $[x, y]_\pm = xy \pm yx$. The remaining (anti)commutators are equal to zero. It is seen that the quantity H is LF Hamiltonian ($H = P_+$). The operator of the momentum P_- is equal to

$$P_- = \int d^2x dx^-\, T_{--} = \frac{1}{2}\int d^2x^\perp \int\limits_{\varepsilon}^{\infty} dk_-\, k_-(a^+(k_-)a(k_-) + a(k_-)a^+(k_-) + \\ +b_r^+(k_-)b_r(k_-) - b_r(k_-)b_r^+(k_-) + c_r^+(k_-)c_r(k_-) - c_r(k_-)c_r^+(k_-)). \tag{43}$$

The quantities $P_+ \equiv H$ and P_- should be normally ordered with respect to creation and annihilation operators. The lowest eigenstate of the momentum P_- is the physical vacuum. It is defined by conditions

$$a(k_-, x^\perp)|\Omega\rangle = 0, \quad b(k_-, x^\perp)|\Omega\rangle = 0, \quad c(k_-, x^\perp)|\Omega\rangle = 0, \quad \forall\, x^\perp,\, k_- \geq \varepsilon > 0. \tag{44}$$

The equalities (8), (9) become true after normal ordering of the operators P_+ and P_-. For the P_- it is seen from the formulae (43), and for the P_+ it follows from the fact that every term of P_+ contains a δ-function of difference between the sum of momenta k_- of creation operators and the sum of momenta k_- of annihilation operators. Due to the positivity of all momenta k_-, in our regularization scheme every term of the P_+ contains at least one annihilation operator. Therefore for normally ordered P_+ we have $P_+|\Omega\rangle = 0$.

The model under consideration requires ultraviolet regularization. It can be done in different ways. We consider this question together with the renormalization problem in sect. 3 and 4.

2.3 The U(N)-theory of pure gauge fields

We consider the $U(N)$ rather than the $SU(N)$ theory because it is technically more simple. The transition to the $SU(N)$ can be done easily. Gauge field is described by Hermitian matrices

$$A_\mu(x) = A_\mu^+(x) \equiv \left\{ A_\mu^{ij}(x) \right\}, \tag{45}$$

where $\mu = +, -, 1, 2$; $i, j = 1, 2, \ldots, N$. Let us assume, that for the indexes i, j and analogous the usual rule of summation on repeated indexes is not used, and where it is necessary the sign of a sum is indicated. Field strengths tensor is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \tag{46}$$

and gauge transformation has the form

$$A_\mu \to A'_\mu = U^+ A_\mu U + \frac{i}{g} U^+ \partial_\mu U, \quad U^+ U = I. \tag{47}$$

To escape a breakdown of gauge invariance we apply the regularization of the type (2) with periodic boundary conditions

$$A_\mu(x^+, -L, x^\perp) = A_\mu(x^+, L, x^\perp), \tag{48}$$

on the interval $-L \leq x^- \leq L$ (this exact periodicity can be always achieved starting from gauge invariant one, i. e. the periodicity up to a gauge transformation). All Fourier modes of $A_\mu(x)$ in x^- must be kept, including zero modes (at $k_- = 0$).

The Lagrangian has the form

$$L = -\frac{1}{2} \int d^2x^\perp \int\limits_{-L}^{L} dx^- \mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu} \right), \tag{49}$$

or

$$L = \int d^2x^\perp \int\limits_{-L}^{L} dx^- \mathrm{Tr}\left(F_{+-}^2 + 2F_{-k}F_{+k} - \frac{1}{2} F_{kk'}F_{kk'} \right), \tag{50}$$

where $k, k' = 1, 2$. Time derivatives $\partial_+ A_k$ are present only in the term

$$\mathrm{Tr}\left(2(\partial_- A_k - \partial_k A_- - ig[A_-, A_+])\partial_+ A_k \right). \tag{51}$$

This expression can be rewritten in standard canonical form only after fixing the gauge in a special form of the type $A_- = 0$ because then the term above becomes similar to that of scalar field theory $\mathrm{Tr}\left(2(\partial_- A_k)\partial_+ A_k\right)$. However not every field, periodic in x^-, can be transformed to the $A_- = 0$ gauge. Indeed, the loop integral

$$\Gamma(x^+, x^\perp) = \mathrm{Tr}\left\{ \mathrm{P}\exp\left(i \int\limits_{-L}^{L} dx^- \, A_-(x^+, x^-, x^\perp) \right) \right\}, \tag{52}$$

where the symbol 'P' means the ordering of operators along the x^-, is a gauge invariant quantity. If this integral is not equal to N for some field then the gauge $A_- = 0$ is not possible for this field. Therefore we choose more weak gauge condition

$$A_-^{ij} = 0, \quad i \neq j, \qquad \partial_- A_-^{ii} = 0, \tag{53}$$

and put

$$A_-^{ii}(x) = v^i(x^+, x^\perp). \tag{54}$$

The gauge (53) breaks not only the local gauge invariance but, unlike the $A_- = 0$ gauge, also global gauge invariance (it remains only the abelian subgroup of gauge transformations not depending on x^-). This has some technical inconvenience but now any periodic field can be described in the gauge (53). Furthermore, if we restrict the class of possible periodic fields by the condition $A_- = 0$, disregarding the described consideration, we come to canonical theory with even more complicate constraints if zero modes are taken into account [21, 22, 24].

From the point of view of LF canonical formalism the variables A_-^{ij} are "coordinates". Therefore one can restrict their values by the condition (53) directly in the Lagrangian without loosing any equations of motion. Let us introduce the denotations

$$D_- A_+^{ij} = (\partial_- - ig(v^i - v^j))A_+^{ij}, \tag{55}$$

$$D_- A_k^{ij} = (\partial_- - ig(v^i - v^j))A_k^{ij}, \tag{56}$$

where obviously

$$D_- A_k^{ii} = \partial_- A_k^{ii}. \tag{57}$$

Also for any function $f(x^-) \equiv f(x^+, x^-, x^\perp)$ periodic in x^- we denote

$$f_{(0)} = \frac{1}{2L}\int\limits_{-L}^{L} dx^-\, f(x^-), \tag{58}$$

$$[f(x^-)] = f(x^-) - f_{(0)}. \tag{59}$$

Obviously,

$$\int\limits_{-L}^{L} dx^-\, [f(x^-)] = 0. \tag{60}$$

After gauge fixing (53) the Lagrangian (50) takes the form

$$L = \int d^2x \int\limits_{-L}^{L} dx^- \Bigg\{ 2\sum_i \left(\partial_- \left[A_k^{ii}\right]\right) \partial_+ \left[A_k^{ii}\right] + 2\sum_{i,j,\, i\neq j} \left(D_- A_k^{ij}\right) \partial^+ A_k^{ji} + \sum_i \left(\partial_- \left[A_+^{ii}\right]\right)^2 +$$

$$+\sum_{i,j,i\neq j} \left(D_- A_+^{ij}\right) D_- A_+^{ji} + 2\sum_i \left[A_+^{ii}\right] \Bigg[\partial_k \partial_- \left[A_k^{ii}\right] - ig \sum_{j',\, j'\neq i} \left(A_k^{ij'} D_- A_k^{j'i} - \left(D_- A_k^{ij'}\right) A_k^{j'i}\right) \Bigg] +$$

$$+2\sum_{i,j,i\neq j} A_+^{ij}\left(\partial_k D_- A_k^{ji} - ig\sum_{j'}\left(A_k^{jj'}D_-A_k^{j'i} - \left(D_-A_k^{jj'}\right)A_k^{j'i}\right)\right) - \frac{1}{2}\sum_{i,j} F_{kl}{}^{ij}F^{klji}\Big\}+$$

$$+\int d^2x\Big\{2L\sum_i(\partial_+v^i)^2 - 4L\sum_i\left(\partial_k A_{k(0)}^{ii}\right)\partial_+v^i - 2\sum_i A_{+(0)}^{ii}\Big(2L\partial_k\partial_k v^i +$$

$$+ig\int\limits_{-L}^{L} dx^-\sum_{j',j'\neq i}\left(A_k^{ij'}D_-A_k^{j'i} - \left(D_-A_k^{ij'}\right)A_k^{j'i}\right)\Big)\Big\}. \quad (61)$$

Here we have ignored some unessential surface terms.

Variation of the Lagrangian in $[A_+^{ii}]$, A_+^{ij} at $i\neq j$ leads to constraints, the solution of which can be written in the form

$$[A_+^{ii}] = \partial_-^{-2}\left[\partial_k\partial_-[A_k^{ii}] - ig\sum_{j',j'\neq i}\left(A_k^{ij'}D_-A_k^{j'i} - (D_-A_k^{ij'})A_k^{j'i}\right)\right], \quad (62)$$

$$A_+^{ij}\Big|_{i\neq j} = D_-^{-2}\left(\partial_k D_-A_k^{ij} - ig\sum_{j'}\left(A_k^{ij'}D_-A_k^{j'j} - (D_-A_k^{ij'})A_k^{j'j}\right)\right). \quad (63)$$

The operator ∂_-^{-1} is completely defined, as before, by the condition (31) being well defined on functions $[f(x)]$. The operator D_-^{-1} after Fourier transformation in x^- is reduced to the multiplication by $\left(i(k_- - g(v^i - v^j))\right)^{-1}$. Therefore it has, in general, no singularities for any $k_- = n(\pi/L)$ with integer n.

Substituting the expressions (62), (63) into the Lagrangian (61), we exclude from it the quantities $[A_+^{ii}]$ and A_+^{ij} at $i\neq j$. The variation of the Lagrangian in $A_{+(0)}^{ii}$ leads to the constraints

$$Q^{ii}(x^+,x^\perp)\equiv -2\left(2L\partial_k\partial_k v^i + ig\int\limits_{-L}^{L}dx^-\sum_{j',j'\neq i}\left(A_k^{ij'}D_-A_k^{j'i} - (D_-A_k^{ij'})A_k^{j'i}\right)\right) = 0. \quad (64)$$

They are first class constraints which can be posed on physical state vectors after quantization. Therefore we can keep the term with this constraint in the Lagrangian.

Now we must put in the standard canonical form the terms of the Lagrangian

$$\int d^2x\int\limits_{-L}^{L}dx^-\left\{2\sum_i(\partial_-[A_k^{ii}])\partial_+[A_k^{ii}] + 2\sum_{i,j,i\neq j}(D_-A_k^{ij})\partial_+A_k^{ji}\right\}. \quad (65)$$

It can be reached by going to Fourier transform

$$[A_k^{ii}(x^-)] = \frac{1}{2\sqrt{2L}}\sum_{k_-=\pi/L}^{\infty} k_-^{-1/2}\left\{a_k^i(k_-)\exp(-ik_-x^-) + a_k^{i\,+}(k_-)\exp(ik_-x^-)\right\}, \quad (66)$$

where we sum over $k_- = n\pi/L$, $n = 1,2,\ldots$, and

$$A_k^{ij}(x^-)\Big|_{i\neq j} = \frac{1}{2\sqrt{2L}}\Big\{\sum_{k_->g(v^i-v^j)}\left(k_- - g(v^i-v^j)\right)^{-1/2}a_k^{ij+}(k_-)\exp(ik_-x^-)+$$

$$+\sum_{k_->g(v^j-v^i)}\left(k_- - g(v^j-v^i)\right)^{-1/2}a_k^{ji}(k_-)\exp(-ik_-x^-)\Big\}, \quad (67)$$

where we sum over all $k_- = n\pi/L$ satisfying corresponding inequalities. The expression (65) takes the form

$$(2i)^{-1}\int d^2x^\perp \left\{ \sum_i \sum_{k_-=\pi/L}^{\infty} \left(a_k^i(k_-)\partial_+(a_k^i)^+(k_-) - a_k^{i\,+}(k_-)\partial_+ a_k^i(k_-) \right) + \right.$$

$$\left. + \sum_{i,j,\, i\neq j} \sum_{k_- > g(v^i - v^j)} \left(a_k^{ij}(k_-)\partial_+ a_k^{ij+}(k_-) - a_k^{ij+}(k_-)\partial_+ a_k^{ij}(k_-) \right) \right\}. \quad (68)$$

Further, in the Lagrangian (61) there is a part

$$L_v = \int d^2x^\perp \left\{ 2L\sum_i (\partial_+ v^i)^2 - 4L \sum_i (\partial_k A^{ii}_{k(0)})\partial_+ v^i \right\}. \quad (69)$$

The "momentum" conjugated to v^i is

$$\mathcal{P}^i = \frac{\delta L_v}{\delta(\partial_+ v^i)} = 4L\left(\partial_+ v^i - \partial_k A^{ii}_{k(0)}\right), \quad (70)$$

Hence,

$$\partial_+ v^i = \frac{1}{4L}\mathcal{P}^i + \partial_k A^{ii}_{k(0)}. \quad (71)$$

The corresponding part of the Hamiltonian equals to

$$H_v = \int d^2x^\perp \sum_i \left(\mathcal{P}^i \partial_+ v^i\right) - L_v = \int d^2x^\perp 2L \sum_i \left(\frac{\mathcal{P}^i}{4L} + \partial_k A^{ii}_{k(0)}\right)^2, \quad (72)$$

and the corresponding part of canonical Lagrangian is

$$L_v = \int d^2x^\perp \sum_i \left(\mathcal{P}^i \partial_+ v^i\right) - H_v. \quad (73)$$

Excluding from the Lagrangian (61) the quantities $[A_+^{ii}]$ and A_+^{ij} (at $i \neq j$) via the equations (66), (67), replacing the terms (65) by the expression (68) and the part (69) by the expression (73), we obtain the result

$$L = (2i)^{-1}\int d^2x^\perp \left\{ \sum_i \sum_{k_-=\pi/L}^{\infty} \left(a_k^i(k_-)\partial_+ a_k^{i\,+}(k_-) - a_k^{i\,+}(k_-)\partial_+ a_k^i(k_-) \right) + \right.$$

$$+ \sum_{i,j,\, i\neq j} \sum_{k_- > g(v^i - v^j)} \left(a_k^{ij}(k_-)\partial_+ a_k^{ij+}(k_-) - a_k^{ij+}(k_-)\partial_+ a_k^{ij}(k_-) \right) +$$

$$\left. + \sum_i \mathcal{P}^i \partial_+ v^i + \sum_i A^{ii}_{+(0)} Q^{ii} \right\} - H, \quad (74)$$

where Q^{ii} are defined by (64), and the Hamiltonian $H = P_+$ is equal to

$$H = \int d^2x \int_{-L}^{L} dx^- \left\{ \sum_i \left(\partial_- \left[A_+^{ii}\right]\right)^2 + \sum_{ij,\, i\neq j} \left(D_- A_+^{ij}\right) D_- A_+^{ji} - \frac{1}{2}\sum_{i,j} F_{kl}{}^{ij} F^{klji} \right\} +$$

$$+2L \int d^2x^\perp \sum_i \left(\frac{\mathcal{P}_i}{4L} + \partial_k A^{ii}_{k(0)}\right)^2. \quad (75)$$

It is implied that instead of the quantities $[A_+^{ii}]$ and A_+^{ij} (at $i \neq j$) one uses the expressions (62), (63) and the A_k^{ij} are expressed in terms of a_k^i, ${a_k^i}^+$, a_k^{ij}, ${a_k^{ij}}^+$, (at $i \neq j$) and of $A_{k(0)}^{ii}$ with the help of equations (66), (67) and

$$A_k^{ii} = \left[A_k^{ii}\right] + A_{k(0)}^{ii}. \tag{76}$$

It is seen from the formulae (74) that ${a_k^i}^+$, a_k^i, ${a_k^{ij}}^+$, a_k^{ij} play the role of creation and annihilation operators. After quantization they satisfy the following commutation relations (at $x^+ = const$):

$$\left[a_k^i(k_-, x^\perp), {a_l^j}^+(k'_-, x'^\perp)\right]_- = \delta^{ij}\delta_{kl}\delta_{k_-,k'_-}\delta^2(x^\perp - x'^\perp), \tag{77}$$

$$\left[a_k^{ij}(k_-, x^\perp), {a_l^{i'j'}}^+(k'_-, x'^\perp)\right]_- = \delta^{ii'}\delta^{jj'}\delta_{kl}\delta_{k_-,k'_-}\delta^2(x^\perp - x'^\perp), \quad i \neq j, \;\; i' \neq j'. \tag{78}$$

Also we have

$$\left[\mathcal{P}^i(x^\perp), v^j(x'^\perp)\right]_- = -i\delta^{ij}\delta^2(x^\perp - x'^\perp). \tag{79}$$

Remaining commutators are equal to zero.

The operator of the momentum P_-, defined by

$$P_- = \int d^2x \int_{-L}^{L} dx^- T_{--}, \tag{80}$$

acts on physical states $|\Psi\rangle$, satisfying the condition

$$Q^{ii}(x^\perp)|\Psi\rangle = 0, \tag{81}$$

equivalent to the canonical operator

$$P_-^{can} = \int d^2x \left(\sum_i \sum_{k_- = \pi/L}^{\infty} k_- {a_k^i}^+(k_-) a_k^i(k_-) + \sum_{i,j,\, i\neq j} \sum_{k_- > g(v^i - v^j)} k_- {a_k^{ij}}^+(k_-) a_k^{ij}(k_-) \right), \tag{82}$$

where the normal ordering was made.

Physical vacuum $|\Omega\rangle$ satisfies the relations

$$a_k^i(k_-, x^\perp)|\Omega\rangle = 0, \tag{83}$$

$$a_k^{ij}(k_-, x^\perp)|\Omega\rangle = 0, \qquad i \neq j, \tag{84}$$

and the condition (81).

This scheme is connected with the following essential difficulty. The zero modes $A_{k(0)}^{ii}(x^\perp)$ are present in the Lagrangian (74) and in the Hamiltonian (75) but the derivatives $\partial_+ A_{k(0)}^{ii}$ are absent there. Therefore new constraints arise

$$\frac{\delta H}{\delta A_{k(0)}^{ii}(x^\perp)} = 0. \tag{85}$$

These constraints are of the 2nd class and they must be solved with respect to $A^{ii}_{k(0)}$ and then the $A^{ii}_{k(0)}$ have to be excluded from the Hamiltonian. The constraints (85) are very complicated and explicit resolution of them is practically impossible. The application of Dirac brackets does not simplify this.

Due to this difficulty a practical calculation usually ignores all zero modes from the beginning. It makes the approximation worse. It is interesting that in the framework of lattice regularization it is possible to overcome the difficulties caused by the constraints (85) [19, 20]. This question will be considered in sect. 5.

3 Limiting transition from the theory in Lorentz coordinates to the theory on the Light Front

To clarify the connection between the theory in Lorentz coordinates in Hamiltonian form and analogous theory on the LF we perform the limiting transition from one to the other. Here this transition is considered in the fixed frame of Lorentz coordinates by introducing states that move at a speed close to the speed of light in the direction of the x^3 axis. Constructing the matrix elements of the Hamiltonian between such states and studying the limiting transition to the speed of light (an infinite momentum), we can derive information about the Hamiltonian in the light-like coordinates. This information also takes into account the contribution from intermediate states with finite momenta. Here, we illustrate the results of such an investigation using (1+1)-dimensional theory of scalar field with the $\lambda\varphi^4$-interaction. Instead of x^3 we denote analogous space coordinate by x^1. The generalization of the method to $(3+1)$-dimensional Yukawa model is discussed briefly at the end of this section. The limiting transition studied here is accomplished approximately by subjecting the momenta p_1 to an auxiliary cutoff that separates fast modes of the fields (with high p_1 values) from slow modes (with finite p_1 values). This cutoff is parameterized in terms of the quantities Λ, Λ_1, and δ and the limiting-transition parameter η ($\eta > 0$, $\eta \to 0$): we have $\eta^{-1}\Lambda_1 \geq |p_1| \geq \eta^{-1}\delta$ for the fast modes and $p_1 \leq \Lambda$ for the slow modes ($\Lambda \gg \delta$). For $\eta \to 0$, the inequality $\eta^{-1}\delta > \Lambda$ holds, so that the above momentum intervals are separated. The field modes with the momenta $\eta^{-1}\delta > |p_1| > \Lambda$ are discarded. This procedure is justified by the fact that the resulting Hamiltonian in the limit $\eta \to 0$ reproduces the canonical LF Hamiltonian (without zero modes) when only the fast modes are taken into account and is consistent with conventional Feynman perturbation theory for $\delta \to 0$. Therefore, even an approximate inclusion of the other (slow) modes may provide a description of nonperturbative effects, such as vacuum condensates. The effective LF Hamiltonian obtained here for the model under consideration differs from the canonical Hamiltonian only by the presence of the vacuum expectation value of the scalar field and by an additional renormalization of the mass of this field. The renormalized mass involves the vacuum expectation value of the squared slow part of the field. Masses of bound states can be found by solving Schrodinger equation

$$P_+|\Psi\rangle = \frac{m^2}{2p_-}|\Psi\rangle, \tag{86}$$

with obtained Hamiltonian P_+.

We start from the standard expression for the Hamiltonian of scalar field $\varphi(x)$ in $(1+1)$-

dimensional space-time in Lorentz coordinates $x^\mu = (x^0, x^1)$, at $x^0 = 0$:

$$H =: \int d^1x \left(\frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_1 \varphi)^2 + \frac{m^2}{2}\varphi^2 + \lambda\varphi^4 \right) :, \tag{87}$$

where $\Pi(x^1)$ are the variables that are canonically conjugate to $\varphi(x^1) \equiv \varphi(x^0 = 0, x^1)$, and the symbol : : of the normal ordering refers to the creation and annihilation operators a and a^+ that diagonalize the free part of the Hamiltonian in the Fock space over the corresponding vacuum $|0\rangle$. These operators are given by

$$\varphi(x^1) = \frac{1}{\sqrt{4\pi}} \int dp_1 (m^2 + p_1^2)^{-1/4} \left[a(p_1) \exp(-ip_1 x^1) + h.c. \right], \tag{88}$$

$$\Pi(x^1) = \frac{-i}{\sqrt{4\pi}} \int dp_1 (m^2 + p_1^2)^{-1/4} \left[a(p_1) \exp(-ip_1 x^1) - h.c. \right], \tag{89}$$

where $a(p_1)|0\rangle = 0$.

To investigate the limiting transition to the LF Hamiltonian (defined at $x^+ = 0$), it is more convenient to go over from the Hamiltonian (87) to the operator $H + P_1 = \sqrt{2}P_+$, where the momentum P_1 has the form

$$P_1 = \int dp_1 a^+(p_1) a(p_1) p_1. \tag{90}$$

Applying the above parametrization of high momenta in terms of $\eta, \eta \to 0$, to p_1 we can then consider the transition to an infinitely high momentum of states as a limit of the corresponding Lorentz transformation with parameter η. To be more specific, we have $p_1 \to (-\eta\sqrt{2})^{-1} q_-$, where q_- is a finite momentum in the light-like coordinates, and

$$\lim_{\eta \to 0} \left((\eta\sqrt{2})^{-1} \langle p_1' | (H + P_1)_{x^0=0} | p_1 \rangle \right) = \langle q_-' | (P_+)_{x^+=0} | q_- \rangle. \tag{91}$$

It follows that the eigenvalues E_+ of the operator $(P_+)_{x^+=0}$ that correspond to the momentum q_- are obtained as the corresponding limit of the eigenvalues $E(\eta)$ of the operator $(H + P_1)_{x^0=0}$ at momentum p_1:

$$E_+ = \lim_{\eta \to 0} (\eta\sqrt{2})^{-1} E(\eta). \tag{92}$$

In the following, we consider this limiting transition as a part of the eigenvalue problem for the operator $H + P_1$, using perturbation theory in the parameter η. Separating the Fourier modes of the field into fast and slow ones, as is indicated above, and neglecting the region of intermediate momenta ($\eta^{-1}\delta \le |p_1| \le \eta^{-1}\Lambda_1$ is the region of the fast modes, and $|p_1| \le \Lambda$ is the region of the slow modes), we can substantially simplify this perturbation theory. The η dependence of the field operators and Hamiltonian can then be determined by making, in the region of fast momenta, the change of the variables as

$$p_1 = \eta^{-1}k, \quad a(p_1) = \sqrt{\eta}\tilde{a}(k), \quad \delta \le |k| \le \Lambda_1, \quad [\tilde{a}(k), \tilde{a}^+(k')] = \delta(k - k'). \tag{93}$$

The fast part $[\varphi(x^1)]_f$ of the operator $\varphi(x^1)$ is estimated as

$$[\varphi(x^1)]_f = \tilde{\varphi}(y) + O(\eta^2), \quad y = \eta^{-1}x^1, \tag{94}$$

$$\tilde{\varphi}(y) = (4\pi)^{-1/2} \int dk|k|^{-1/2}[\tilde{a}(k)\exp(-iky) + h.c.]. \quad (95)$$

We denote the slow part of the field φ by $\breve{\varphi}$ ($\varphi = [\varphi]_f + \breve{\varphi}$). Substituting formulas (93)-(95) into Hamiltonian (87), we obtain

$$H + P_1 = \eta^{-1}\left(h_0 + \eta h_1 + \eta^2 h_2 + \ldots\right), \quad (96)$$

$$h_0 = 2\int\limits_{\delta}^{\Lambda_1} dk\tilde{a}^+(k)\tilde{a}(k)k, \quad (97)$$

$$h_1 = (H + P_1)_{\varphi=\breve{\varphi},\Pi=\breve{\Pi}} \equiv (\breve{H} + \breve{P}_1), \quad (98)$$

$$h_2 = \int\limits_{\delta\le|k|\le\Lambda_1} dk\left(\frac{m^2}{2|k|}\right)\tilde{a}^+(k)\tilde{a}(k) + :\lambda\int dy\left[\tilde{\varphi}^4(y) + 4\breve{\varphi}(0)\tilde{\varphi}^3(y) + 6\breve{\varphi}^2(0)\tilde{\varphi}^2(y)\right]: . \quad (99)$$

Prior to performing integration with respect to y, we formally expanded the operators $\breve{\varphi}(x^1) = \breve{\varphi}(\eta y)$ in Taylor series in the variable ηy and estimated their orders in the parameter η at fixed y. Such estimates can be justified at least in Feynman perturbation theory. The operator $(\breve{H} + \breve{P}_1)$ in (98) is defined in such a way that its minimum eigenvalue is zero. Let us consider perturbation theory in the parameter η for the equation

$$(H + P_1)|f\rangle = E|f\rangle \quad (100)$$

under the condition that the states $|f\rangle$ have, as in formula (91), a negative value of P_1 proportional to η^{-1} for $\eta \to 0$ and also describe the states with a finite mass. The expansions of the quantity E and the vector $|f\rangle$ in power series in η can then be written as

$$E = \eta^{-1}\sum_{n=2}^{\infty}\eta^n E_n, \qquad |f\rangle = \sum_{n=0}^{\infty}\eta^n|f_n\rangle. \quad (101)$$

We arrive at the system of equations

$$h_0|f_0\rangle = 0, \quad (102)$$

$$h_0|f_1\rangle + h_1|f_0\rangle = 0, \quad (103)$$

$$h_0|f_2\rangle + h_1|f_1\rangle + (h_2 - E_2)|f_0\rangle = 0, \quad \ldots. \quad (104)$$

To describe solutions to these equations, we use the basis generated by the fast-field operators $\tilde{a}^+(k)$ over the vacuum $|0\rangle$ and the slow-field operators $\breve{\varphi}$ and $\breve{\Pi}$ over the vacuum $|v\rangle$ that corresponds to the Hamiltonian $(\breve{H} + \breve{P}_1)$. The vectors of this basis can be symbolically represented as

$$\tilde{a}^+\ldots\tilde{a}^+|0\rangle\breve{\varphi}\ldots\breve{\varphi}\breve{\Pi}\ldots\breve{\Pi}|v\rangle. \quad (105)$$

By virtue of (97), the manifold of solutions $|f_0\rangle$ to equation (102) is reduced to the set of vectors (105), which do not contain the operators $\tilde{a}^+(k)$ with $k \geq \delta$. Let $\mathcal{P}_0$ be the projection operator onto this set. According to equation (103), we then have

$$\mathcal{P}_0 h_0 |f_1\rangle = (\breve{H} + \breve{P}_1)|f_0\rangle = 0. \tag{106}$$

Equation (106) requires that the vectors $|f_0\rangle$ be the lowest eigenstates of the operator $(\breve{H}+\breve{P}_1)$, that is, linear combinations of basis vectors (105) including neither the operators $\tilde{a}^+(k)$ with $k \geq \delta$ nor the operators $\breve{\varphi}$ and $\breve{\Pi}$. We denote the projection operator on this set of vectors (105) by $\mathcal{P}'_0$. To determine the quantity E_2 which we are interested in, it is sufficient to consider the $\mathcal{P}'_0$-projection of equation (104). Taking into account (102), (105), and (106), we find that E_2 appears as a solution to the eigenvalue problem

$$\mathcal{P}'_0 h_2 |f_0\rangle = E_2 |f_0\rangle. \tag{107}$$

Thus, in accordance with (91) and (92), the operator $\mathcal{P}'_0 h_2 \mathcal{P}'_0$ plays the role of the effective LF Hamiltonian P_+^{eff}. Substituting formula (99) for the operator h_2 into the expression for P_+^{eff}, we take into account that, between the projection operators $\mathcal{P}'_0$, the contribution of the field modes with positive momenta ($k \geq \delta$) vanishes and that the products of the operators of the slow part of the field can be replaced with their expectation values for the vacuum $|v\rangle$. In addition, we note that, under the Lorentz transformation corresponding to the limiting transition $\eta \to 0$ in formula (91), the variable y goes over into the light-like coordinate $y^- = -y/\sqrt{2}$, the momenta k go over into the light-like momenta $q_- = -\sqrt{2}k$, and the corresponding coordinate y^+ vanishes at $x^0 = 0$ (for finite values of y^-). Going over to the operators $A(q_-) = 2^{-1/4}\tilde{a}(-k)$ for $k \leq -\delta$ ($q_- \geq \delta\sqrt{2}$) and to the corresponding field $\Phi(y^+ = 0, y^-) = \tilde{\varphi}_-(y)$, where $\tilde{\varphi}_-$ is the part of the $\tilde{\varphi}$ containing only the modes with negative momenta ($k \leq -\delta$), we obtain the effective Hamiltonian P_+^{eff} in the form

$$P_+^{eff} =: \int dy^- \left\{ \frac{1}{2}\left[m^2 + 12\lambda \langle : \breve{\varphi}^2 : \rangle_v \right] \Phi^2 + 4\lambda \langle \breve{\varphi} \rangle_v \Phi^3 + \lambda \Phi^4 \right\} :, \tag{108}$$

where $\langle \ldots \rangle_v$ is the expectation value for the vacuum $|v\rangle$. Expression (108) coincides with the canonical effective Hamiltonian if, in the latter, we take into account the shift of the field by the constant $\langle \breve{\varphi} \rangle_v$ and the change in the mass squared by $12\lambda \left[\langle : \breve{\varphi}^2 : \rangle_v - \langle \breve{\varphi} \rangle_v^2 \right]$.

Analogous results were obtained for Yukawa model in $(3+1)$-dimensional space-time [8] in regularization of Pauli-Villars type, introducing a number of nonphysical fields with very large masses. The absence of essential difference between the Hamiltonian obtained via limiting transition and canonical LF Hamiltonian is connected with this choice of regularization. Other regularizations can lead to more complicated results.

This method of limiting transition can not be directly expanded to gauge theories, because the approximations used for nongauge theories are not justified.

4 Comparison of Light Front perturbation theory with the theory in Lorentz coordinates

As is already known, canonical quantization in LF, i.e., on the $x^+ = const$ hypersurface, can result in a theory not quite equivalent to the Lorentz-invariant theory (i.e., to the standard Feynman formalism). This is due, first of all, to strong singularities at zero values of the "light-like"

momentum variables $Q_- = \frac{1}{\sqrt{2}}(Q_0 - Q_3)$. To restore the equivalence with a Lorentz-covariant theory, one has to add unusual counter-terms to the formal canonical Hamiltonian for the LF, $H = P_+$. These counter-terms can be found by comparing the perturbation theory based on the canonical LF formalism with Lorentz-covariant perturbation theory [11]. This is done in the present section. The LF Hamiltonian thus obtained can then be used in nonperturbative calculations. It is possible, however, that perturbation theory does not provide all of the necessary additions to the canonical Hamiltonian, as some of these additions can be nonperturbative. In spite of this, it seems necessary to examine this problem within the framework of perturbation theory first.

For practical purposes a stationary noncovariant LF perturbation theory, which is similar to the one applied in nonrelativistic quantum mechanics, is widely used. It was found [25–27] that the "light-front" Dyson formalism allows this theory to be transformed into an equivalent LF Feynman theory (under an appropriate regularization). Then, by re-summing the integrands of the Feynman integrals, one can recast their form so that they become the same as in the Lorentz-covariant theory. (This is not the case for diagrams without external lines, which we do not consider here.) Then, the difference between the LF and Lorentz-covariant approaches that persists is only due to the different regularizations and different methods of calculating the Feynman integrals (which is important because of the possible absence of their absolute convergence in pseudo-Euclidean space). In the present section, we concentrate on the analysis of this difference.

A LF theory needs not only the standard UV regularization, but also a special regularization of the singularities $Q_- = 0$. In our approach, this regularization (by method (1)) eliminates the creation operators $a^+(Q)$ and annihilation operators $a(Q)$ with $|Q_-^i| < \varepsilon$ from the Fourier expansion of the field operators in the field representation. As a result, the integration w.r.t. the corresponding momentum Q_- over the range $(-\infty, -\varepsilon) \cup (\varepsilon, \infty)$ is associated with each line before removing the δ-functions. Different propagators are regularized independently, which allows the described re-arrangement of the perturbation theory series. On the other hand, this regularization is convenient for further nonperturbative numerical calculations with the LF Hamiltonian, to which the necessary counter-terms are added (the "effective" Hamiltonian). We require that this Hamiltonian generate a theory equivalent to the Lorentz-covariant theory when the regularization is removed. Note that Lorentz-invariant methods of regularization (e.g., Pauli-Villars regularization) are far less convenient for numerical calculations and we shall only briefly mention them.

The specific properties of the LF Feynman formalism manifest themselves only in the integration over the variables $Q_\pm = \frac{1}{\sqrt{2}}(Q_0 \pm Q_3)$, while integration over the transverse momenta $Q_\perp \equiv \{Q_1, Q_2\}$ is the same in the LF and the Lorentz coordinates (though it might be nontrivial because it requires regularization and renormalization). Therefore, we concentrate on a comparison of diagrams for fixed transverse momenta (which is equivalent to a two-dimensional problem).

In this section we propose a method that allows one to find the difference (in the limit $\varepsilon \to 0$) between any LF Feynman integral and the corresponding Lorentz-covariant integral without having to calculate them completely. Based on this method, a procedure is elaborated for constructing an effective LF Hamiltonian correct to any order of perturbation theory. This procedure can be applied to nongauge field theories as well as to Abelian and non-Abelian gauge theories in the gauge $A_- = 0$ with the gauge vector field propagator chosen according to the Mandelstam-Leibbrandt prescription [28, 29]. The question of whether the additions

to the Hamiltonian, that arise under that procedure, can be combined into a finite number of counter-terms must be considered separately in each particular case.

We will consider the application of this formalism to Yukawa model, to QCD in four-dimensional space-time and to QED in two-dimensional space-time.

4.1 Reduction of Light Front and Lorentz-covariant Feynman integrals to a form convenient for comparison

Let us examine an arbitrary one particle irreducible Feynman diagram. We fix all external momenta and all transverse momenta of integration, and integrate only over Q_+ and Q_-:

$$F = \lim_{æ\to 0} \int \frac{\prod_i d^2Q^i \quad f(Q^i, p^k)}{\prod_i (2Q^i_+ Q^i_- - M_i^2 + iæ)}. \tag{109}$$

We assume that all vertices are polynomial and that the propagator has the form

$$\frac{z(Q)}{Q^2 - m^2 + iæ}, \qquad \text{or} \qquad \frac{z(Q)\; Q_+}{(Q^2 - m^2 + iæ)(2Q_+Q_- + iæ)}, \tag{110}$$

where $z(Q)$ is a polynomial. A propagator of the second type in (110) arises in gauge theories in the gauge $A_- = 0$ if the Mandelstam-Leibbrandt formalism [28, 29] with the vector field propagator

$$\frac{1}{Q^2 + iæ}\left(g_{\mu\nu} - \frac{Q_\mu \delta^+_\nu + Q_\nu \delta^+_\mu}{2Q_+Q_- + iæ} 2Q_+\right), \tag{111}$$

is used. In equation (109) either $M_i^2 = m_i^2 + {Q^i_\perp}^2 \neq 0$, where m_i is the particle mass, or $M_i^2 = 0$.

The function f involves the numerators of all propagators and all vertices with the necessary δ-functions, that include the external momenta p^k (the same expression without the δ-functions is a polynomial, which we denote by $\tilde{f}$). We assume for the diagram F and for all of its subdiagrams that the conditions

$$\omega_\parallel < 0, \qquad \omega_+ < 0, \tag{112}$$

hold, where ω_+ is the index of divergence w.r.t. Q_+ at $Q^i_- \neq 0\ \forall i$, and $\omega_\parallel$ is the index of divergence in Q_+ and Q_- (simultaneously); $Q_\pm = \frac{1}{\sqrt{2}}(Q_0 \pm Q_3)$. The diagrams that do not meet these conditions should be examined separately for each particular theory (their number is usually finite). We seek the difference between the value of integral (109) obtained by the Lorentz-covariant calculation and its value calculated in LF coordinates (LF calculation).

In the LF calculation, one introduces and then removes the LF cutoff $|Q_-| \geq \varepsilon > 0$:

$$F_{\rm lf} = \lim_{\varepsilon\to 0} \lim_{æ\to 0} \int\limits_{V_\varepsilon} \prod_i dQ^i_- \int \prod_i dQ^i_+ \frac{f(Q^i, p^k)}{\prod_i (2Q^i_+ Q^i_- - M_i^2 + iæ)}, \tag{113}$$

where $V_\varepsilon = \prod_i ((-\infty, -\varepsilon) \cup (\varepsilon, \infty))$. Here (and in the diagram configurations to be defined below) we take the limit w.r.t. ε, but, generally speaking, this limit may not exist. In this case, we assume that we do not take the limit, but take the sum of all nonpositive power terms of

the Laurent series in ε at the zero point. If conditions (112) are satisfied, Statement 2 from Appendix I can be used. This results in the equality

$$F_{\mathrm{lf}} = \lim_{\varepsilon\to 0}\lim_{æ\to 0}\int\prod_k dq_+^k \int\limits_{V_\varepsilon\cap B_L}\prod_k dq_-^k \frac{\tilde{f}(Q^i,p^s)}{\prod_i(2Q_+^iQ_-^i - M_i^2 + iæ)}. \tag{114}$$

From here on, the momenta of the lines Q^i are assumed to be expressed in terms of the loop momenta q^k, B_L is a sphere of a radius L in the q_-^k-space, and L depends on the external momenta. Now, using Statement 2 from Appendix I, we obtain

$$F_{\mathrm{lf}} = \lim_{\varepsilon\to 0}\lim_{æ\to 0}\lim_{\beta\to 0}\lim_{\gamma\to 0}\int\prod_k dq_+^k \int\limits_{V_\varepsilon}\prod_k dq_-^k \frac{\tilde{f}(Q^i,p^s)\; e^{-\gamma\sum_i {Q_+^i}^2 - \beta\sum_i {Q_-^i}^2}}{\prod_i(2Q_+^iQ_-^i - M_i^2 + iæ)}. \tag{115}$$

To reduce the covariant Feynman integral to a form similar to (114), we introduce a quantity $\hat{F}$:

$$\hat{F} = \lim_{æ\to 0}\lim_{\beta\to 0}\lim_{\gamma\to 0}\int\prod_k d^2q^k \frac{\tilde{f}(Q^i,p^s)\; e^{-\gamma\sum_i {Q_+^i}^2 - \beta\sum_i {Q_-^i}^2}}{\prod_i(2Q_+^iQ_-^i - M_i^2 + iæ)}. \tag{116}$$

Let us prove that this quantity coincides with the result of the Lorentz-covariant calculation F_{cov}. To this end, we introduce the α-representation in the Minkowski space of the propagator

$$\frac{z(Q^i)}{2Q_+^iQ_-^i - M_i^2 + iæ} = -iz\left(-i\frac{\partial}{\partial y_i}\right)\int\limits_0^\infty e^{i\alpha_i(2Q_+^iQ_-^i - M_i^2 + iæ) + i(Q_+^i y_i^+ + Q_-^i y_i^-)}d\alpha_i\Big|_{y_i=0}. \tag{117}$$

Then we substitute (117) into (116). Due to the exponentials that cut off q_+^k, q_-^k and α^i the integral over these variables is absolutely convergent. Therefore, one can interchange the integrations over q_+^k, q_-^k and α^i. As a result, we obtain the equality

$$\hat{F} = \lim_{æ\to 0}\lim_{\beta\to 0}\lim_{\gamma\to 0}\int\limits_0^\infty\prod_n d\alpha_i\;\hat{\varphi}(\alpha_i,p^s,\gamma,\beta)\; e^{-æ\sum_i\alpha_i}, \tag{118}$$

where

$$\begin{aligned}\hat{\varphi}(\alpha_i,p^s,\gamma,\beta) = (-i)^n\tilde{f}\left(-i\frac{\partial}{\partial y_i}\right)\times&\\ \times\int\prod_k d^2q^k\; e^{\sum_i\left[i\alpha_i(2Q_+^iQ_-^i - M_i^2) + i(Q_+^i y_i^+ + Q_-^i y_i^-) - \gamma {Q_+^i}^2 - \beta {Q_-^i}^2\right]}\Big|_{y_i=0}&.\end{aligned} \tag{119}$$

For the Lorentz-covariant calculation in the α-representation satisfying conditions (112), there is a known expression [31]

$$F_{\mathrm{cov}} = \lim_{æ\to 0}\int\limits_0^\infty\prod_n d\alpha_i\;\varphi_{\mathrm{cov}}(\alpha_i,p^s)\; e^{-æ\sum_i\alpha_i}, \tag{120}$$

where

$$\begin{aligned}\varphi_{\mathrm{cov}}(\alpha_i,p^s) = (-i)^n\tilde{f}\left(-i\frac{\partial}{\partial y_i}\right)\times&\\ \times\lim_{\gamma,\beta\to 0}\int\prod_k d^2q^k\; e^{\sum_i\left[i\alpha_i(2Q_+^iQ_-^i - M_i^2) + i(Q_+^i y_i^+ + Q_-^i y_i^-) - \gamma {Q_+^i}^2 - \beta {Q_-^i}^2\right]}\Big|_{y_i=0}&.\end{aligned} \tag{121}$$

In Appendix 2, it is shown that in (118) the limits in γ and β can be interchanged, in turn, with the integration over $\{\alpha_i\}$, and then with $\tilde{f}\left(-i\frac{\partial}{\partial y_i}\right)$. After that, a comparison of relations (118), (119) and (120), (121), clearly shows that $\hat{F} = F_{\text{cov}}$. Considering (116) and using Statement 1 from Appendix 1, we obtain the equality

$$F_{\text{cov}} = \lim_{\text{æ}\to 0} \int \prod_k dq_+^k \int_{B_L} \prod_k dq_-^k \frac{\tilde{f}(Q^i, p^s)}{\prod_i (2Q_+^i Q_-^i - M_i^2 + i\text{æ})}. \tag{122}$$

Expression (122) differs from (114) only by the range of the integration over q_-^k.

4.2 Reduction of the difference between the Light Front and Lorentz-covariant Feynman integrals to a sum of configurations

Let us introduce a partition for each line,

$$\left(\int_{-\infty}^{-\varepsilon} dQ_- + \int_{\varepsilon}^{\infty} dQ_-\right) = \left[\int dQ_- + (-1)\int_{-\varepsilon}^{\varepsilon} dQ_-\right]. \tag{123}$$

We call a line with integration w.r.t. the momentum Q_-^i in the range $(-\varepsilon, \varepsilon)$ (before removing the δ-functions) a type-1 line, a line with integration in the range $(-\infty, -\varepsilon) \cup (\varepsilon, \infty)$ a type-2 line, and a line with integration over the whole range $(-\infty, \infty)$ a full line. In the diagrams, they are denoted as shown in Figs. la, b, and c, respectively.

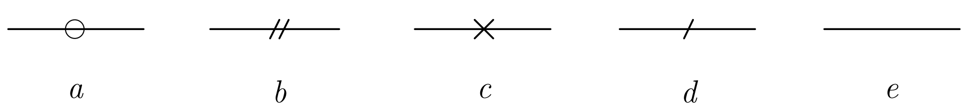

Figure 1. Notation for different types of lines in the diagrams: "a" is a type-1 line, "b" is a type-2 line, "c" is a full line, "d" is an ε-line, and "e" is a Π-line.

Let us substitute partition (123) into expression (114) for F_{lf} and open the brackets. Among the resulting terms, there is F_{cov} (expression (122)). We call the remaining terms "diagram configurations" and denote them by F_j. Then we arrive at the relation $F_{\text{lf}} - F_{\text{cov}} = \sum_j F_j$, where

$$F_j = \lim_{\varepsilon\to 0} \lim_{\text{æ}\to 0} \int \prod_k dq_+^k \int_{V_\varepsilon^j \cap B_L} \prod_k dq_-^k \frac{\tilde{f}(Q^i, p^s)}{\prod_i (2Q_+^i Q_-^i - M_i^2 + i\text{æ})}, \tag{124}$$

and V_ε^j is the region corresponding to the arrangement of full lines and type-1 lines in the given configuration.

Note that before taking the limit in ε, equations (122) and (124) can be used successfully: first, they are applied to a subdiagram and, then, are substituted into the formula for the entire diagram. This is admissible because, after the deformation of the contours described in the

proof of Statement 1 from Appendix 1, the integral over the loop momenta $\{q_+^k\}$ of the subdiagram converges (after integration over the variables $\{q_-^k\}$ of this subdiagram) absolutely and uniformly with respect to the remaining loop momenta $\{q_-^{k'}\}$. Therefore, one can interchange the integrals over $\{q_+^k\}$ and $\{q_-^{k'}\}$.

Thus, the difference between the LF and Lorentz-covariant calculations of the diagram is given by the sum of all of its configurations. A configuration of a diagram is the same diagram, but where each line is labeled as a full or type-1 line, provided that at least one type-1 line exists.

4.3 Behavior of the configuration as $\varepsilon \to 0$

We assume that all external momenta p^s are fixed for the diagram in question and

$$p_-^s \neq 0, \quad \sum_{s'} p_-^{s'} \neq 0, \tag{125}$$

where the summation is taken over any subset of external momenta; all of these momenta are assumed to be directed inward.

Let us consider an arbitrary configuration. We apply the term "ε-line" to all type-1 lines and those full lines for which integration over Q_- actually does not expand outside the domain $(-r\varepsilon, r\varepsilon)$, where r is a finite number (below, we explain when these lines appear). The remaining full lines are called Π-lines. In the diagrams, the ε-lines and Π-lines are denoted as shown in Figs. 1d and e, respectively. Note that the diagram can be drawn with lines "a" and "c" from Fig. 1 (this defines the configuration unambiguously), or with lines "d" and "e" (then the configuration is not uniquely defined).

If among the lines arriving at the vertex only one is full and the others are type-1 lines, this full line is an ε-line by virtue of the momentum conservation at the vertex. The remaining full lines form a subdiagram (probably unconnected). By virtue of conditions (125), there is a connected part to which all of the external lines are attached. All of the external lines of the remaining connected parts are ε-lines. Consequently, using Statement 1 from Appendix 1, we can see that integration over the internal momenta of these connected parts can be carried out in a domain of order ε in size, i.e., all of their internal lines are ε-lines. Thus, an arbitrary configuration can be drawn as in Fig. 2 and integral (124), with the corresponding integration domain, is associated with it.

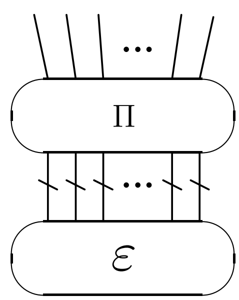

Figure 2. Form of an arbitrary configuration: Π is the connected subdiagram consisting of Π-lines, ℰ is the subdiagram consisting of ε-lines and, probably, containing no vertices.

Let us investigate the behavior of the configuration as $\varepsilon \to 0$. From here on, it is convenient to represent the propagator as

$$\frac{\tilde{z}(Q)}{Q^2 - m^2 + i\ae}, \quad \text{where} \quad \tilde{z}(Q) = z(Q) \ \text{ or } \ \tilde{z}(Q) = \frac{z(Q)}{2Q_- + i\ae/Q_+}. \tag{126}$$

rather than as (110). Then, in (109), $M_i^2 = m_i^2 + {Q_\perp^i}^2 \neq 0$ and the function $\tilde{f}$ is no longer a polynomial. If the numerator of the integrand consists of several terms, we consider each term separately (except when the terms arise from expressing the propagator momentum Q_-^i in terms of loop and external momenta).

We denote the loop momenta of subdiagram Π in Fig. 2 by q^l and the others by k^m. We make following change of integration variables in (124):

$$k_-^m \to \varepsilon\, k_-^m. \tag{127}$$

Then, the integration over k_-^m goes within finite limits independent of ε. We denote the power of ε in the common factor by τ (it stems from the volume elements and the numerators when the transformation (127) is made). The contribution to τ from the expression $1/(2Q_- + i\ae/Q_+)$ (equation (126)), which is related to the ε-line, is equal to -1. We divide the domain of integration over k_+^m and q_+^l into sectors such that the momenta of all full lines Q_+^i have the same sign within one sector.

In Statement 1 of Appendix 1, it is shown that for each sector, the contours of integration over q_-^l and k_-^m can be bent in such a way that absolute convergence in q_+^l, k_+^m, q_-^l and k_-^m takes place. Since, in this case, the momenta Q_-^i of Π-lines are separated from zero by an ε-independent constant, the corresponding Π-line-related propagators and factors from the vertices can be expanded in a series in ε. This expansion commutes with integration.

It is also clear that the denominators of the propagators allow the following estimates under an infinite increase in $|Q_+|$:

$$\left|\frac{1}{2Q_+Q_- - M^2 + i\ae}\right| \leq \begin{cases} \dfrac{1}{c\,|Q_+|} & \text{for } \Pi\text{-lines}, \quad (128) \\[2ex] \dfrac{1}{\tilde{c}\,\varepsilon\,|Q_+|} & \text{for } \varepsilon\text{-lines}, \quad (129) \end{cases}$$

Here c and $\tilde{c}$ are ε-independent constants. Note that for fixed finite Q_+, the estimated expressions are bounded as $\varepsilon \to 0$. After transformation (127) and release of the factor $\frac{1}{\varepsilon}$ (in accordance with what was said about the contribution to τ), the ε-line-related expression from (126) becomes

$$\left|\frac{1}{2Q_- + i\ae/Q_+}\right| \to \left|\frac{1}{2Q_- + i\ae/(Q_+\varepsilon)}\right| \leq \frac{1}{2|Q_-|}, \tag{130}$$

where a Q_+-independent quantity was used for the estimate (this quantity is meaningful and does not depend on ε because the value of Q_- is separated from zero by an ε-independent constant).

We integrate first over q_+^l, k_+^m within one sector and then over q_-^l, k_-^m (the latter integral converges uniformly in ε). Let us examine the convergence of the integral over q_+^l, k_+^m with canceled denominators of the ε-lines (which is equivalent to estimating the expressions (129)

by a constant). If it converges, then the initial integral is obviously independent of ε and the contribution from this sector to the configuration is proportional to ε^{τ}.

Let us show that if it diverges with a degree of divergence α, the contribution to the initial integral is proportional to $\varepsilon^{\tau-\alpha}$ up to logarithmic corrections. To this end, we divide the domain of integration over q_+^l, k_+^m into two regions: U_1, which lies inside a sphere of radius Λ/ε (Λ is fixed), and U_2, which lies outside this sphere (recall that in our reasoning, we deal with each sector separately). Now we estimate (128) (like (129)) in terms of $\dfrac{1}{\hat{c}\varepsilon|Q_+|}$ (which is admissible) and change the integration variables as follows:

$$q_+^l \to \frac{1}{\varepsilon}\, q_+^l, \quad k_+^m \to \frac{1}{\varepsilon}\, k_+^m. \tag{131}$$

After ε is factored out of the numerator and the volume element, the integrand becomes independent of ε. Thus, the integral converges.

One can choose such Λ (independent of ε) that the contribution from the domain U_2 is smaller in absolute value than the contribution from the domain U_1. Consequently, the whole integral can he estimated via the integral over the finite domain U_1. Now we make an inverse replacement in (131) and estimate (129) by a constant (as above). Since the size of the integration domain is Λ/ε and the degree of divergence is α, the integral behaves as $\varepsilon^{-\alpha}$ (up to logarithmic corrections), q.e.d. This reasoning is valid for each sector and, thus, for the configuration as a whole. Obviously,

$$\alpha = \max_r \alpha_r, \tag{132}$$

where α_r is the subdiagram divergence index and the maximum is taken over all subdiagrams D_r (including unconnected subdiagrams for which α_r is the sum of the divergence indices of their connected parts). In the case under consideration, $\alpha_r = \omega_+^r + \nu^r$, where ν^r is the number of internal ε-lines in the subdiagram D_r. The quantities $\omega_\pm^r$ are the UV divergence indices of the subdiagram D_r w.r.t. $Q_\pm$.

Above, we introduced a quantity τ, which is equal to the power of ε that stems from the numerators and volume elements of the entire configuration. We can write $\tau = \omega_-^r - \mu^r + \nu^r + \eta^r$, where μ^r is the index of the UV divergence in Q_- of a smaller subdiagram (probably, a tree subdiagram or a nonconnected one) consisting of Π-lines entering D_r. The term η^r is the power of ε in the common factor, which, during transformation (127), stems from the volume elements and numerators of the lines that did not enter D_r. (It is implied that the integration momenta are chosen in the same way as when calculating the divergence indices of D_r.) Then, up to logarithmic corrections, we have

$$F_j \sim \varepsilon^{\sigma}, \quad \sigma = \min_r(\tau, \omega_-^r - \omega_+^r - \mu^r + \eta^r). \tag{133}$$

Consequently, for $\varepsilon \to 0$, the configuration is equal to zero if $\sigma > 0$. Relation (133) allows all essential configurations to be distinguished.

4.4 Correction procedure and analysis of counter-terms

We want to build a corrected LF Hamiltonian $H_{\rm lf}^{\rm cor}$ with the cutoff $|Q_-^i| > \varepsilon$, which would generate Green's functions that coincide in the limit $\varepsilon \to 0$ with covariant Green's functions within

the perturbation theory. We begin with a usual canonical Hamiltonian in the LF coordinates $H_{\rm lf}$ with the cutoff $|Q^i_-| > \varepsilon$. We imply that the integrands of the Feynman diagrams derived from this LF Hamiltonian coincide with the covariant integrands after some resummation [25–27]. However, a difference may arise due to the various methods of doing the integration, e.g., due to different auxiliary regularizations. As shown in Sec. 4.2, this difference (in the limit $\varepsilon \to 0$) is equal to the sum of all properly arranged configurations of the diagram. One should add such correcting counter-terms to $H_{\rm lf}$, which generates additional "counter-term" diagrams, that reproduce nonzero (after taking limit w.r.t. ε) configurations of all of the diagrams. Were we able to do this, we would obtain the desired $H^{\rm cor}_{\rm lf}$. In fact, we can only show how to seek the $H^{\rm cor}_{\rm lf}$ that generates the Green's functions coinciding with the covariant ones everywhere except the null set in the external momentum space (defined by condition (125)). However, this restriction is not essential because this possible difference does not affect the physical results.

Our correction procedure is similar to the renormalization procedure. We assume that the perturbation theory parameter is the number of loops. We carry out the correction by steps: first, we find the counterterms to the Hamiltonian that generate all nonzero configurations of the diagrams up to the given order and, then, pass to the next order. We take into account that this step involves the counter-term diagrams that arose from the counter-terms added to the Hamiltonian for lower orders. Thus, at each step, we introduce new correcting counter-terms that generate the difference remaining in this order. Let us show how to successfully look for the correcting counter-terms.

We call a configuration nonzero if it does not vanish as $\varepsilon \to 0$. We call a nonzero configuration "primary" if Π is a tree subdiagram in it (see Fig. 2). Note that for this configuration, breaking any Π-line results in a violation of conditions (125); then, the resulting diagram is not a configuration. We say that the configuration is changed if all of the Π-lines in the related integral (124) are expanded in series in ε (see the reasoning above equation (128) in Sec. 4.3) and only those terms that do not vanish in the limit $\varepsilon \to 0$ after the integration are retained. As mentioned above, developing this series and integration are interchangeable operations. Thus, in the limit $\varepsilon \to 0$, the changed and unchanged configurations coincide. Therefore, we always require that the Hamiltonian counter-terms generate changed configurations, as this simplifies the form of the counter-terms. Using additional terms in the Hamiltonian, one can generate only counter-term diagrams, which are equal to zero for external momenta meeting the condition $|p^s_-| < \varepsilon$, because with the cutoff used, the external lines of the diagrams do not carry momenta with $|p^s_-| < \varepsilon$. We bear this in mind in what follows.

We seek counter-terms by the induction method. It is clear that, in the first order in the number of loops, all nonzero configurations are primary. We add the counter-terms that generate them to the Hamiltonian. Now, we examine an arbitrary order of perturbation theory. We assume that in lower orders, all nonzero configurations that can be derived from the counter-terms, accounting for the above comment, have already been generated by the Hamiltonian.

Let us proceed to the order in question. First, we examine nonzero configurations with only one loop momentum k and a number of momenta q (see the notation above equation (127)). We break the configuration lines one by one without touching the other lines (so that the ends of the broken lines become external lines). The line break may result in a structure that is not a configuration (if conditions (125) are violated); a line break may also result in a zero configuration or in a nonzero configuration. If the first case is realized for each broken line, then the initial configuration is primary and it must be generated by the counter-terms of the Hamiltonian in the order under consideration. If breaking of each line results in either the first

or the second case, we call the initial configuration real and it must be also generated in this order.

Assume that breaking a line results in the third case. This means that the resulting configuration stems from counter-terms in the lower orders. Then, after restoration of the broken line (i.e., after the appropriate integration), it turns out that the counter-terms of the lower orders have generated the initial configuration (we take into account the comment on successive application of equation (124); see the end of Sec. 4.2) with the following distinctions: (i) the broken line (and, probably, some others, if a nonsimply connected diagram arises after breaking the line) is not a Π-line but a type-2 line, due to the conditions $|p^s_-| > \varepsilon$; (ii) if, after restoration of the broken line, the behavior at small ε becomes worse (i.e., σ decreased), then fewer terms than are necessary for the initial configuration were considered in the above-mentioned series in ε. We expand these arising type-2 lines by formula (123) and obtain a term where all of these lines are replaced by Π-lines or other terms where some (or all) of these lines have become type-1 lines. In the latter case, one of the momenta q becomes the momentum k. We call these terms "repeated parts of the configuration" and analyze them together with the configurations that have two momenta k. In the former case, we obtain the initial configuration up to distinction (ii). We add a counter-term to the Hamiltonian that compensates this distinction (the counter-term diagrams generated by it are called the compensating diagrams).

If there is only one line for which the third case is realized, it turns out that, in the given order, it is not necessary to generate the initial configuration by the counter-terms, except for the compensating addition and the repeated part that is considered at the next step. If there are several lines for which the third case is realized, the initial configuration is generated in lower orders more than once. For compensation, it should be generated (with the corresponding numerical coefficient and the opposite sign) by the Hamiltonian counter-terms in the given order. We call this configuration a secondary one. Next, we proceed to examine configurations with two momenta k and so on up to configurations with all momenta k, which are primary configurations.

Thus, the configurations to be generated by the Hamiltonian counter-terms can be primary (not only the initial primary configurations but also the repeated parts analogous to them, called primary-like), real, compensating, and secondary. If the theory does not produce either the loop consisting only of lines with Q_+ in the numerator (accounting for contributions from the vertices) or a line with ${Q_+}^n$ in the numerator for $n > 1$, then real configurations are absent because a line without Q_+ in the numerator can always be broken without increasing σ (see equation (133)). It is not difficult to demonstrate that if each appearing primary, real, and compensating configuration has only two external line, then there are no secondary configurations at all.

The dependence of the primary configuration on external momenta becomes trivial if its degree of divergence α is positive, the maximum in formula (132) is reached on the diagram itself, and $\sigma = 0$. Then, only the first term is taken into account in the above-mentioned series. Thus, not all of the Π-line-related propagators and vertex factors depend on k^m_- and they can be pulled out of the sign of the integral w.r.t. $\{k^m_-\}$ in (124). We then obtain

$$F_j^{\mathrm{prim}} = \lim_{\varepsilon\to 0}\lim_{æ\to 0}\int\prod_m dk^m_+ \frac{\tilde{f}'(k^m,p^s)}{\prod_i(2Q^i_+Q^i_- - M_i^2 + iæ)}\times$$

$$\times \int_{V_\varepsilon} \prod_m dk_-^m \frac{\tilde{f}''(k^m)}{\prod_k (2Q_+^k Q_-^k - M_k^2 + i\text{æ})}, \quad (134)$$

where V_ε is a domain of order ε in size. Let us carry out transformations (127) and (131). For the denominator of the Π-line, we obtain

$$\frac{1}{2(\frac{1}{\varepsilon}\sum k_+ + \sum p_+)(\sum p_-) - M^2 + i\text{æ}} \to \frac{\varepsilon}{2(\sum k_+)(\sum p_-)}. \quad (135)$$

Here we neglect terms of order ε in the denominator because the singularity at $k_+^m = 0$ is integrable under the given conditions for α and everything can be calculated in zero order in ε at $\sigma = 0$. Thus, the dependence on external momenta can be completely collected into an easily obtained common factor.

4.5 Application to the Yukawa model

The Yukawa model involves diagrams that do not satisfy condition (112). These are displayed in Figs. 3a and b. We have $\omega_\parallel = 0$ for diagram "a" and $\omega_+ = 0$ for diagram "b".

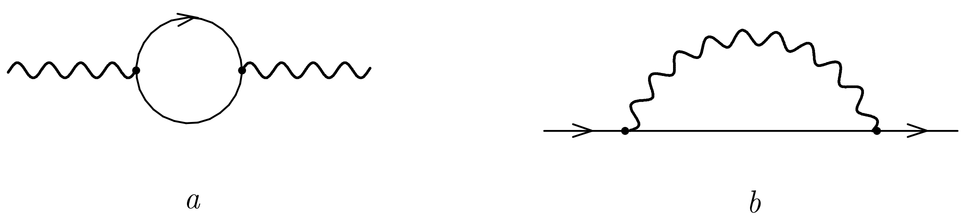

Figure 3. Yukawa model diagrams that do not meet condition (112).

Nevertheless, these diagrams can be easily included in the general scheme of reasoning. To this end, one should subtract the divergent part, independent of external momenta, in the integrand of the logarithmically divergent (in two-dimensional space, with fixed internal transverse momenta) diagram "a". We obtain an expression with $\omega_\parallel < 0$ (i.e., which converges in two-dimensional space) and $\omega_+ = 0$, as in diagram "b". This means that the integral over q_+ converges only in the sense of the principal value (and it is this value of the integral that should be taken in the LF coordinates to ensure agreement with the stationary noncovariant perturbation theory). This value can be obtained by distinguishing the q_+-even part of the integrand.

Two approaches are possible. One is to introduce an appropriate regularization in transverse momenta and to imply integration over them; then, it is convenient to distinguish the part that is even in four-dimensional momenta q. The other is to keep all transverse momenta fixed; then, the part that is even in longitudinal momenta $q_\parallel$ can be released. For the Yukawa theory, we use the first approach. For the transverse regularization, we use a "smearing" of vertices, which is equivalent to dividing each propagator by $1 + {Q_\perp^i}^2/{\Lambda_\perp}^2$. In four-dimensional space, diagram "a" diverges quadratically. Under introduction and subsequent removal of the transverse regularization, the divergent part, which was previously subtracted from this diagram, acquires the form $C_1 + C_2\, p_\perp^2$.

After separating the even part of the regularized expression, we fix all of the transverse momenta again. Then it turns out that diagrams "a" and "b" in Fig. 3 meet conditions (112) and one can show that after all of the operations mentioned, the exponent σ (see (133)) does not decrease for any of their configurations. Hence, they can be included in the general scheme without any additional corrections.

Let us first analyze the primary configurations (see the definition in Sec. 4.4). In the numerators, k_- appears only in the zero or one power and there are no loops where the numerators of all of the lines contain k_-. Consequently, one always has $\tau > 0$, $\mu^r \leq 0$, and $\eta^r \geq 0$ (see the definitions in Sec. 4.3). Analyzing the properties of the expression $\omega_-^r - \omega_+^r$ for the Yukawa model diagrams, we conclude from (133) that $\sigma \geq 0$ always holds. The general form of the nonzero primary configurations with $\sigma = 0$ is depicted in Fig. 4. Note that they are all configurations with two external line.

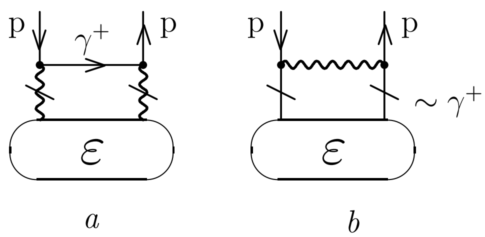

Figure 4. Nonzero configurations in the Yukawa model: p is the external momentum, and γ^+ or γ^- symbols on the line indicate that the corresponding term is taken in the numerator of the propagator. In configuration "b", the part that is proportional to γ^+ is taken.

Further, it is clear that there are no nonzero real configurations (see the comment at the end of Sec. 4.4), and it can be shown by induction that there are no nonzero compensating or secondary configurations either (the definitions are given in Sec. 4.4 also). Thus, only primary or primary-like configurations can be nonzero and all of them have the form shown in Fig. 4. It can be shown that their degree of divergence α is positive and the maximum in formula (132) is reached for the diagram itself. Thus, the reasoning above and below formula (134) applies to them. Then, denoting the configurations displayed in Figs. 4a-d by $D_a - D_d$, we arrive at the equalities $D_a = \frac{\gamma^+}{p_-} C_a$, $D_b = \frac{\gamma^+}{p_-} C_b$, $D_c = C_c$ and $D_d = C_d$, where the expressions $C_a - C_d$ depend only on the masses and transverse momenta, but not on the external longitudinal momenta, and have a finite limit as $\varepsilon \to 0$.

Now we assume that $D_a - D_d$ are not single configurations but are the sums of all configurations of the same form and that integration over the internal transverse momenta has already been carried out, (with the above-described regularization). In four-dimensional space, the diagrams D_a and D_b diverge linearly while D_c and D_d diverge quadratically. Therefore, because of the transverse regularization, the coefficients C_c and C_d in the limit of removing this regularization take the form $C_1 + C_2\, p_\perp^2$, where C_1 and C_2 do not depend on the external momenta (neither do C_a, C_b)). Thus, to generate all nonzero configurations by the LF Hamiltonian, only the expression

$$H_c = \tilde{C}_1\, \varphi^2 + \tilde{C}_2\, p_\perp^2\, \varphi^2 + \tilde{C}_3\, \bar{\psi}\, \frac{\gamma^+}{p_-}\, \psi, \tag{136}$$

should be added, where φ and ψ are the boson and fermion fields, respectively, and $\tilde{C}_i$, are the constant coefficients.

Comparing (136) with the initial canonical LF Hamiltonian, one can easily see that the found counter-terms are reduced to a renormalization of various terms of the Hamiltonian (in particular the boson mass squared and the fermion mass squared without changing the term linear in fermion mass). The explicit Lorentz invariance is absent but results are Lorentz invariant because the counterterms compensate the violation of Lorentz invariance inherent to chosen LF formalism.

Note that the second approach, mentioned at the beginning of this section, can give the same results. The only difference is that in two-dimensional space, the contributions from the configurations displayed in Fig. 3 would additionally depend on external transverse momenta. However, this dependence disappears after integration over internal transverse momenta with the introduction and subsequent removal of an appropriate regularization.

In the Pauli Villars regularization, it is easy to verify that the expression $\omega^r_- - \omega^r_+ - \mu^r + \eta^r$ from (133) increases. This is because the number of terms in the numerators of the propagator increases. Then, the contribution from the ε-lines does not change, while the Π-lines belonging to D_r make zero contribution to $\omega^r_- - \omega^r_+$ and η^r, but -1 contribution to μ^r. Since $\tau > 0$, this regularization makes it possible to meet the condition $\sigma > 0$ for the configurations that were nonzero (one additional boson field and one additional fermion field are enough). Then it turns out that the canonical LF Hamiltonian need not be corrected at all.

Obtained results agree with the conclusions of the work [25], where a comparison of LF and Lorentz-covariant methods was made for self-energy diagrams in all orders of perturbation theory and for other diagrams in lowest orders.

4.6 Application to QCD(3+1)

Applying the LF Hamiltonian approach to gauge theories under regularization (1), where zero modes of fields are thrown out, one has to use the gauge $A_- = 0$ (see, for example, sect. 2.3). To carry out successfully renormalization procedure for this scheme it is necessary to take gauge field propagator in accordance with Mandelstam-Leibbrandt prescription [28, 29] (which allows to perform Euclidean continuation, see [32–34]). Such a propagator has the form

$$\frac{-i\delta^{ab}}{Q^2 + i\text{æ}}\left(g_{\mu\nu} - \frac{Q_\mu\delta^+_\nu + Q_\nu\delta^+_\mu}{2Q_+Q_- + i\text{æ}}2Q_+\right) = \frac{-i\delta^{ab}}{Q^2 + i\text{æ}}\left(g_{\mu\nu} - \frac{Q_\mu n_\nu + Q_\nu n_\mu}{2(Qn^*)(Qn) + i\text{æ}}2(Qn^*)\right), \qquad (137)$$

where $n_+ = 1$, $n_- = n_\perp = 0$, $n^*_- = 1$, $n^*_+ = n^*_\perp = 0$.

The formalism described in sect. 4.1-4.4 was such that it could be applied to a theory with the propagator (137) (at fixed transverse momenta $Q_\perp \neq 0$). It turns out that there are nonzero configurations with arbitrarily large numbers of external lines. An example of such a configuration is given in Fig. 5.

Indeed, using formula (133), we can see that for the configuration in Fig. 5, $\tau = 0$ and, thus, $\sigma \leq 0$, i.e., this is a nonzero configuration. It is also clear that introduction of the Pauli-Villars regularization does not improve the situation because it does not affect τ.

The difficulty is that the distortion of the pole in (137) due to LF cutoff $|Q_-| \geq \varepsilon > 0$ does not disappear in the limit $\varepsilon \to 0$, and infinite number of new counterterms are required to compensate this distortion [11]. The simplest way to avoid this difficulty is to add small

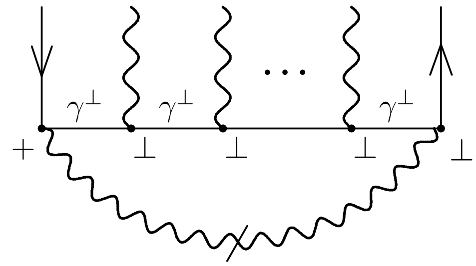

Figure 5. Nonzero configuration with an arbitrarily large number of external lines in a gauge theory. The symbols $\gamma^\perp$ on the lines and the symbols $+$ or $\perp$ the vertices indicate that the corresponding terms γ^+ or $\gamma^\perp$ are taken in the numerators of propagators and in the vertex factors.

mass-like parameter μ^2 in the denominator:

$$\frac{1}{2Q_+Q_- + i\text{æ}} \longrightarrow \frac{1}{2Q_+Q_- - \mu^2 + i\text{æ}}, \tag{138}$$

and take the limit $\varepsilon \to 0$ before $\mu \to 0$). To describe this modification with local Lagrangian we need to introduce ghost fields A'_μ in addition to conventional A_μ. We write the free part of pure gluon Lagrangian as follows (using higher derivatives and the parameter Λ for UV regularization):

$$L_0 = -\frac{1}{4}\left(f^{a,\mu\nu}\left(1+\frac{\partial^2}{\Lambda^2}\right)f^a_{\mu\nu} - f'^{a,\mu\nu}\left(1+\frac{\partial^2}{\Lambda^2}+\frac{2\partial_+\partial_-}{\mu^2}\right)f'^a_{\mu\nu}\right), \tag{139}$$

where $f^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu$, $f'^a_{\mu\nu} = \partial_\mu A'^a_\nu - \partial_\nu A'^a_\mu$ and $A^a_- = A'^a_- = 0$. Interaction terms depend only on summary field $\bar{A}^a_\mu = A^a_\mu + A'^a_\mu$.

At fixed μ and Λ we get a theory with broken gauge invariance but with preserved global $SU(3)$-invariance. We put into the Lagrangian all necessary interaction terms (but with unknown coefficients) including those that are needed for UV renormalization:

$$\begin{aligned} L = L_0 &+ c_0\partial_\mu\bar{A}^a_\nu\partial^\mu\bar{A}^{a,\nu} + c_{01}\partial_\mu\bar{A}^a_\nu n^\mu n^{*\alpha}\partial^\alpha\bar{A}^{a,\nu} + c_1\partial_\mu\bar{A}^{a,\mu}\,\partial_\nu\bar{A}^{a,\nu} + \\ &+c_{11}n^\mu n^{*\alpha}\partial_\mu\bar{A}^{a,\alpha}\,\partial_\nu\bar{A}^{a,\nu} + c_{12}n^\mu n^{*\alpha}\partial_\mu\bar{A}^{a,\alpha}n^\nu n^{*\beta}\partial_\nu\bar{A}^a_\beta + c_2\bar{A}^a_\mu\bar{A}^{a,\mu} + \\ &+c_3 f^{abc}\bar{A}^a_\mu\bar{A}^b_\nu\partial^\mu\bar{A}^{c,\nu} + c_{31}f^{abc}\bar{A}^a_\mu\bar{A}^b_\nu n^\alpha n^{*\mu}\partial^\alpha\bar{A}^{c,\nu} + \\ &+\bar{A}^a_\mu\bar{A}^b_\nu\bar{A}^c_\gamma\bar{A}^d_\delta(c_4 f^{abe}f^{cde}g^{\mu\gamma}g^{\nu\delta} + \delta^{ab}\delta^{cd}\left(c_5 g^{\mu\gamma}g^{\nu\delta} + c_6 g^{\mu\nu}g^{\gamma\delta})\right). \end{aligned} \tag{140}$$

These terms are local and can be taken in Lorentz covariant form due to the restoring of this symmetry in the $\varepsilon \to 0$, $\mu \to 0$ limit. However in constructing of these terms both vectors n^μ and $n^{*\nu}$, included in the definition of the propagator (137), can participate.

For this theory one can apply the formalism described in sect. 4.1-4.4 to compare (at $\varepsilon \to 0$) the LF perturbation theory and that taken in Lorentz coordinates within the same regularization scheme. We find that the difference between mentioned perturbation theories can be compensated by changing of the value of coefficient c_2 before the term of gluon mass form $\bar{A}^a_\mu\bar{A}^{\mu,a}$ in naive LF Hamiltonian of this theory. After that we can analyse further our regularized theory in Lorentz coordinates and even make Euclidean continuation.

It is possible to prove by induction to all orders of perturbation theory that in the limit $\mu \to 0$, $\Lambda \to \infty$ our theory can be made finite and coinciding with the usual renormalized (dimensionally regularized) theory in Light Cone gauge [32, 33] (for all Green functions). This was done in the paper [14], but there the terms of the Lagrangian, including the vector $n^{*\mu}$, were missed. Right expression for LF Hamiltonian should correctly take into account the contribution of all terms, written in the (140).

The values of the unknown coefficients c_i before all counterterms in (140) must be chosen so that the Green functions in each order coincided (after removing the regularization) with those obtained in conventional dimensionally regularized formulation and therefore satisfied Ward identities. Besides, we need to correlate the limits $\mu \to 0$ and $\Lambda \to \infty$ to avoid infrared divergencies at $\mu \to 0$. It is sufficient to take $\mu = \mu(\Lambda)$ and to require that $\mu\Lambda \to 0$ and $(\log \mu)/\Lambda \to 0$.

Our resulting LF Hamiltonian for pure $SU(3)$ gluon fields contains 11 unknown coefficients, including coefficient before gluon mass term that takes also into account the difference between LF and Lorentz coordinate formulations of our regularized theory. The generalization of our scheme for full QCD with fermions is described in [14]. In this case there are 20 unknown coefficients in the LF Hamiltonian. We hope that it is possible to find an analog of Ward identities relating the coefficients c_i at fixed Λ. This problem seems very important for our approach.

4.7 Application to QED(1+1)

The procedure, described in previous section, allows to construct LF Hamiltonian for QCD in four-dimensional space-time, but such LF Hamiltonian contains many additional fields and unknown coefficients. This complicates calculations with this Hamiltonian. Moreover, because only the perturbation theory with respect to the coupling constant was analyzed, there could remain purely nonperturbative effects that are not taken into consideration. It is therefore useful to consider an example of "nonperturbative" (with respect to usual coupling constant) construction of LF Hamiltonian for gauge theory, which is possible for two-dimensional quantum electrodynamics (QED(1+1)), i.e. for "massive" Schwinger model.

The QED(1+1), defined originally by the Lagrangian

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\gamma^m D_\mu - M)\Psi, \tag{141}$$

can be transformed to its bosonized form [35, 36], described by scalar field Lagrangian

$$L = \frac{1}{2}\left(\partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2\right) + \frac{M m e^C}{2\pi} \cos(\theta + \sqrt{4\pi}\Phi), \tag{142}$$

where $m = e/\sqrt{\pi}$ is the Schwinger boson mass (the e is original coupling), $C = 0.577216\ldots$ is the Euler constant, and the θ is the "θ"-vacuum parameter, which takes into account the nontriviality of QED(1+1) quantum vacuum due to instantons. Here the fermion mass M plays the role of the coupling in bosonized theory so that perturbation theory in this coupling corresponds to chiral perturbation theory in QED(1+1). The nonpolynomial form of scalar field interaction leads in perturbation theory to infinite sums of diagrams in each finite order. It can be proved [16, 37] that some partial sums of these infinite sums are UV divergent in the 2nd order, whereas for full (Lorentz-covariant) Green functions these divergencies cancel

(remaining only the divergent vacuum diagrams). Therefore physical quantities are UV finite in this theory. Only at intermediate steps of our analysis we need some UV regularization.

We compare LF and Lorentz-covariant perturbation theories for such bosonized model using an effective resummation of perturbation series in coordinate representation for Feynman diagrams [16, 17] and also using the formalism described in sect. 4.1-4.4. The results of this comparison can be formulated as follows.

The difference between considered perturbation theories can be eliminated in the limit of removing regularizations if we use instead of the naive LF Hamiltonian

$$H = \int dx^- \left(\frac{1}{8\pi} m^2 : \varphi^2 : - \frac{\gamma}{2} e^{i\theta} : e^{i\varphi} : - \frac{\gamma}{2} e^{-i\theta} : e^{-i\varphi} : \right),$$
$$\gamma = \frac{Mme^C}{2\pi}, \quad \varphi = \sqrt{4\pi}\,\Phi, \quad |p_-| \geq \varepsilon > 0, \tag{143}$$

the "corrected" LF Hamiltonian:

$$H = \int dx^- \left(\frac{1}{8\pi} m^2 : \varphi^2 : - B : e^{i\varphi} : - B^* : e^{-i\varphi} : \right) -$$
$$-2\pi e^{-2C} \frac{|B|^2}{m^2} \int dx^- dy^- \left(: e^{i\varphi(x^-)} e^{-i\varphi(y^-)} : -1 \right) \theta(|x^- - y^-| - \alpha) \frac{v(\varepsilon(x^- - y^-))}{|x^- - y^-|}. \tag{144}$$

Here the terms, linear in B and B^* (new coupling constants), are of the same form as in naive Hamiltonian; only the term, containing the $|B|^2$, is of new form (nonlocal in x^-). The α is the UV regularization parameter, and the $v(z)$ is some arbitrary continuous function rapidly decreasing at the infinity and going to unity as $z \to 0$. The coupling B can be perturbatively written as a series in γ:

$$B = \frac{\gamma}{2} e^{i\theta} + \sum_{n=2}^{\infty} \gamma^n B_n. \tag{145}$$

On the other side, it is related to the sum of all connected "generalized tadpole" diagrams (i.e. diagrams with external lines attached to only one vertex), which is described by the "condensate" parameter $A = \frac{\gamma}{2} \langle \Omega | : e^{i(\varphi+\theta)} : | \Omega \rangle$ of the Lorentz-covariant formulation (the $|\Omega\rangle$ is the physical vacuum state in this formulation):

$$B + |B|^2 w = A, \tag{146}$$

$$w = \frac{2\pi e^{-2C}}{m^2} \int dx^- \frac{\theta(|x^-| - \varepsilon\alpha)}{|x^-|} v(x^-). \tag{147}$$

The equation (146) can be solved with respect to the B:

$$B = -\frac{1}{2w} + \sqrt{\frac{1}{4w^2} + \frac{A'}{w} - A''^2} + iA'', \tag{148}$$

where $A = A' + iA''$, and the sign before the root respects the perturbation theory. Within the perturbation theory in γ one cannot remove UV regularization (i. e. to put $\alpha \to 0$ and therefore $w \to \infty$) in this expression due to UV divergencies of the coefficients B_n. However,

taking into account the validity of the equation (146) to all orders in γ, we can consider it beyond the perturbation theory. Then we use the estimation for the A at $\alpha \to 0$ [16]:

$$A = \frac{\gamma^2}{4} w + const \tag{149}$$

and get for the B in $\alpha \to 0$ limit UV finite result:

$$B = \mathrm{sign}(\cos\theta)\sqrt{\frac{\gamma^2}{4} - A''^2} + iA'' = \frac{\gamma}{2} e^{i\hat{\theta}}, \tag{150}$$

so that all information about the condensate is contained in the phase factor $e^{i\hat{\theta}}$:

$$\sin\hat{\theta} = 2\frac{\mathrm{Im}A}{\gamma} = \langle\Omega| : \sin(\varphi + \theta) : |\Omega\rangle. \tag{151}$$

Then we can make a transformation, inverse to the bosonization, but on the LF. Actually we need the expression only for one independent component ψ_+ of the bispinor field $\binom{\psi_+}{\psi_-}$ due to the LF constraint, permitting to write the ψ_- in terms of ψ_+. One can use the exact expression for the ψ_+ in terms of the φ obtained in the theory on the interval $|x^-| \le L$ with periodic boundary conditions [36, 16]. We need only to modify our corrected bosonized theory by using discretized LF momentum p_- instead of continuous one and hence replacing the cutoff parameter ε by π/L. The necessary formulae for the ψ_+ has the following form [16, 36] (we choose here antiperiodic boundary conditions for the fermion fields):

$$\psi_+(x) = \frac{1}{\sqrt{2L}} e^{-i\omega} e^{-i\frac{\pi}{L}x^- Q} e^{i\frac{\pi}{2L}x^-} : e^{-i\varphi(x)} : . \tag{152}$$

The operator ω and the charge operator Q are canonically conjugated so that the ψ_+ defined by the equation (152) has proper commutation relation with the charge. On the other side the operator $e^{i\omega}$ shifts Fourier modes ψ_n of the field ψ_+ [16, 36]:

$$e^{i\omega}\psi_n e^{-i\omega} = \psi_{n+1}. \tag{153}$$

If we separate the modes related with creation and annihilation operators on the LF putting

$$\psi_+(x) = \frac{1}{\sqrt{2L}}\left(\sum_{n\ge 1} b_n e^{-i\frac{\pi}{L}(n-\frac{1}{2})x^-} + \sum_{n\ge 0} d_n^+ e^{i\frac{\pi}{L}(n+\frac{1}{2})x^-}\right), \quad b_n|0\rangle = d_n|0\rangle = 0, \tag{154}$$

we can define the operator $e^{i\omega}$ uniquely by specifying its action on the LF vacuum $|0\rangle$ as follows:

$$e^{i\omega}|0\rangle = b_1^+|0\rangle, \qquad e^{-i\omega}|0\rangle = d_0^+|0\rangle. \tag{155}$$

In such sense this operator is similar to a fermion.

We can now rewrite our corrected boson LF Hamiltonian in terms of ψ_+ and $e^{i\omega}$. The result is remarkably simple:

$$H = \int_{-L}^{L} dx^- \left(\frac{e^2}{2}\left(\partial_-^{-1}[\psi_+^+\psi_+]\right)^2 - \frac{iM^2}{2}\psi_+^+\partial_-^{-1}\psi_+ - \left(\frac{Me\, e^C}{4\pi^{3/2}} e^{-i\hat{\theta}}\, e^{i\omega} d_0^+ + h.c.\right)\right). \tag{156}$$

This fermionic LF Hamiltonian differs from canonical one (in corresponding DLCQ scheme) only by last term, depending on zero modes and vacuum condensate parameter $\hat{\theta}$ which can be related to chiral condensate by transforming the variables in the equation (151):

$$\sin\hat{\theta} = -\frac{2\pi^{3/2}}{e\, e^{C}}\langle\Omega| : \bar{\Psi}\gamma^5\Psi : |\Omega\rangle. \tag{157}$$

Let us remark that the presents of linear in M term in LF Hamiltonian (156) can be considered as a results of a modification of the LF constraint, connecting the ψ_- with the ψ_+. An analogous modification of this constraint was got in the paper [38] where the method of exact operator solution of massless Schwinger model was applied.

The constructed LF Hamiltonian (156) was applied to nonperturbative numerical calculation of mass spectrum of QED(1+1) [18]. The results of this calculation were compared with those of lattice calculations in Lorentz coordinates [39, 40].

The calculations were carried out in wide domain of values of fermion mass M for all values of the parameter $\hat{\theta}$, which is a function of the M/e and vacuum parameter θ. We do not know exactly this function, but know that it must be zero at $\theta = 0$ and be equal to π at $\theta = \pi$.

For $\theta = 0$ the obtained spectrum is bounded from below at any values M, and is in good agreement with the results of the paper [39] for two bound states of lowest mass.

For the value $\theta = \pi$, at which phase transition can take place, the obtained spectrum for the lowest bound state agrees well with the results of the paper [40] for sufficiently small M; at greater M we start to see the disagreement with that paper, and then at M, greater some critical value, the spectrum becomes unbounded from below. This critical value approximately coincides with the point of phase transition found in [40]. It can be supposed that at M greater than the point of phase transition the calculations with our LF Hamiltonian, constructed via the analysis of perturbation theory in M, become incorrect.

Our calculations show that LF Hamiltonian (156) can give good results only in the limited domain of the parameters: from perturbative domain to some values which define the limits of applicability of our LF Hamiltonian. In the domain where our Hamiltonian becomes incorrect we need to take into account nonperturbative (in M/e) effects. Such an investigation can be useful for finding a LF Hamiltonian approach to realistic gauge theories.

The LF Hamiltonian (156) includes the operator ω, which has no simple expression in terms of field operators. It is defined only by its properties (153),(155). Due to this fact the expression for the Hamiltonian depends essentially on the form of the regularization, i.e. $|x^-| \le L$ and antiperiodic boundary conditions in x^- for the field ψ. Now we have found a possibility to rewrite the expression (156) in such a way that it contains only fermion field operators, and describes in the limit of removing the regularizations the same theory as the Hamiltonian (156). This new expression has at $\hat{\theta} = \theta = 0$ the following form:

$$H = \int\limits_{-L}^{L} dx^- \left(\frac{e^2}{2}\left(\partial_-^{-1}[\psi^+\psi]\right)^2 + \frac{eMe^C}{4\pi^{3/2}}\left(d_0^+ d_0 + b_1^+ b_1\right) - \frac{iM^2}{2}\psi^+\partial_-^{-1}\psi \right). \tag{158}$$

Preliminary calculations of the mass spectrum, produced by this Hamiltonian, show that in the limit of removing the regularization, $L \to \infty$, results indeed coinside, with a good accuracy, with those for the bound state mass spectrum found here for the Hamiltonian (156). Work on studing of the Hamiltonian (158) spectrum and also on the constructing the analogous expression for the Hamiltonian at $\hat{\theta} \neq 0$ will be continued in future.

5 Transverse lattice regularization of Gauge Theories on the Light Front

The introduction of space-time lattice for gauge-invariant regularization of nonabelian gauge theories is well known [41]. Gauge invariant regularization in continuous space-time is also known [42] but it seems not suitable for the LF quantization. For the LF formulation only the lattice in transverse coordinates x^1, x^2 is used. In this formulation it is convenient to define variables so as to have the action polynomial in these variables [43, 44, 19, 20]. Such a regularization is not Lorentz invariant, and one can only hope that Lorentz invariance can be restored in continuous space limit. Nevertheless many attempts to apply LF Hamiltonian formulation with periodic boundary conditions in x^-, combined with transverse space lattice, are undertaken (for "color dielectric" type models [45–48]). In all of these works zero modes of fields are thrown out, so that, in fact, gauge invariance is violated.

We consider canonical LF formulation of gauge theories, regularized in gauge-invariant way. To achieve this goal we introduce transverse space lattice, discretize the momentum p_- according to the prescription (b) (see introduction, equation (2)) with all zero modes of fields included and apply the so called "finite mode" ultraviolet regularization in p_-. The last means a cutoff in eigenvalues of covariant derivative operator D_- in the expansion of lattice field variables in eigen functions of this operator. These field variables are lattice modification of transverse components of usual gauge fields. They are described by complex matrices, defined on lattice links. Only such variables admit mentioned above "finite mode" regularization (for fermion fields analogous method was applied in [49, 50]).

It is interesting that in the framework of this formulation one can avoid complicated canonical 2nd-class constraints, usually present in canonical LF formalism in continuous space. This greatly simplifies canonical quantization procedure. However the absence of explicit Lorentz invariance of the regularization scheme makes the investigation of the connection with conventional Lorentz-covariant formulation difficult. In particular, there is a problem of the description of quantum vacuum as common lowest eigen state of both operators P_- and P_+.

5.1 Gauge-invariant action on the transverse lattice

At first we introduce particular ultraviolet regularization via a lattice in transverse coordinates x^1, x^2 and choose variables so as to have the action, which is polynomial in these variables [43, 44]. Furthermore, we use the described in the introduction gauge-invariant regularization (b) (see equation (2)) of singularities at $p_- \to 0$ and gauge-invariant ultraviolet cutoff in modes of covariant derivative D_- (then ultraviolet regularization of the theory is complete). For simplicity further we consider again the $U(N)$ theory of pure gauge fields because this example is technically more simple than the $SU(N)$ theory.

The components of gauge field along continuous coordinates x^+, x^- can be taken without a modification and related to the sites of the lattice. Transverse components are described by complex $N \times N$ matrices $M_k(x)$, $k = 1, 2$. Each matrix $M_k(x)$ is related to the link directed from the site $x - e_k$ to the site x. The transverse vector e_k connects two neighbouring sites on the lattice being directed along the positive axis x^k ($|e_k| \equiv a$), see fig. 6. The matrix $M_k^+(x)$ is related to the same link but with opposite direction, see fig. 7. In the following the usual rule of summation over repeated indices is not used for the index k. If this summation is necessary, the symbol of a sum is indicated.

Figure 6.

Figure 7.

The elements of these matrices are considered as independent variables. This makes the action polynomial. For any closed directed loop in the lattice we can construct the trace of the product of matrices $M_k(x)$ sitting on the links in the loop and order these matrices from the right to the left along this loop. For example the expression

$$\mathrm{Tr}\ \left\{M_2(x)M_1(x-e_2)M_2^+(x-e_1)M_1^+(x)\right\} \tag{159}$$

is related to the loop shown in fig. 8.

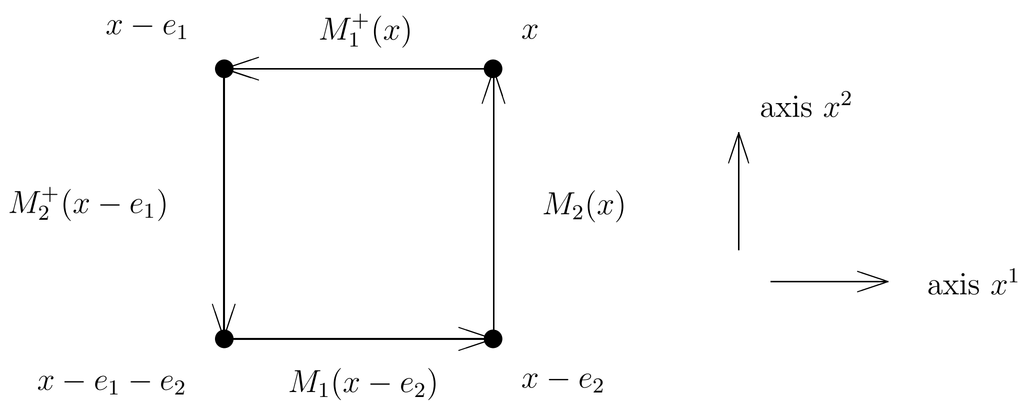

Figure 8.

It should be noticed that a product of matrices related to closed loop, consisting of one and the same link passed in both directions, is not identically unity because the matrices M_k are not unitary (see, for example, fig. 9).

The unitary matrices $U(x)$ of gauge transformations act on the M and M^+ in the following way:

$$M_k(x) \to M_k'(x) = U(x)M_k(x)U^+(x-e_k), \tag{160}$$

$$M_k^+(x) \to M_k'^+(x) = U(x-e_k)M_k^+(x)U^+(x). \tag{161}$$

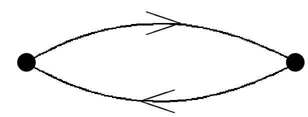

Figure 9.

A trace of the product of the matrices, related to closed loop along lattice links, is invariant with respect to these transformations. To relate the matrices M_k with usual gauge fields of continuum theory let us write these matrices in the following form:

$$M_k(x) = I + gaB_k(x) + igaA_k(x), \qquad B_k^+ = B_k, \quad A_k^+ = A_k. \tag{162}$$

Then in the $a \to 0$ limit the fields $A_k(x)$ coincide with transverse gauge field components, and the $B_k(x)$ turn out to be extra (nonphysical) fields which should be switched off in the limit. Below we show how to get this.

The analog of the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu], \tag{163}$$

multiplied by i, can be defined as follows:

$$\begin{aligned} G_{+-} = iF_{+-}, \qquad F_{+-}(x) &= \partial_+ A_-(x) - \partial_- A_+(x) - ig[A_+(x), A_-(x)], \\ G_{\pm,k}(x) &= \frac{1}{ga}\left[\partial_\pm M_k(x) - ig\left(A_\pm(x)M_k(x) - M_k(x)A_\pm(x-e_k)\right)\right], \\ G_{12}(x) &= -\frac{1}{ga^2}\left[M_1(x)M_2(x-e_1) - M_2(x)M_1(x-e_2)\right]. \end{aligned} \tag{164}$$

Under gauge transformation these quantities transform as follows:

$$\begin{aligned} G_{+-}(x) &\to G'_{+-}(x) = U(x)G_{+-}(x)U^+(x), \\ G_{\pm,k}(x) &\to G'_{\pm,k}(x) = U(x)G_{\pm,k}(x)U^+(x-e_k), \\ G_{12}(x) &\to G'_{12}(x) = U(x)G_{12}(x)U^+(x-e_1-e_2). \end{aligned} \tag{165}$$

We choose a simplest form of the action having correct naive continuum limit:

$$S = a^2 \sum_{x^\perp} \int dx^+ \int_{-L}^{L} dx^- \ \mathrm{Tr}\left[G_{+-}^+ G_{+-} + \sum_k \left(G_{+k}^+ G_{-k} + G_{-k}^+ G_{+k}\right) - G_{12}^+ G_{12}\right] + S_m, \tag{166}$$

where the additional term S_m gives an infinite mass to extra fields B_k in the $a \to 0$ limit:

$$\begin{aligned} S_m = -\frac{m^2(a)}{4g^2} \sum_{x_\perp} \int dx^+ \int_{-L}^{L} dx^- \sum_k \mathrm{Tr}\left[\left(M_k^+(x)M_k(x) - I\right)^2\right] \underset{a\to 0}{\longrightarrow} \\ \underset{a\to 0}{\longrightarrow} -m^2(a) \int d^2x^\perp \int dx^+ \int_{-L}^{L} dx^- \sum_k \mathrm{Tr}\left(B_k^2\right), \qquad m(a) \underset{a\to 0}{\longrightarrow} \infty. \end{aligned} \tag{167}$$

It is supposed that this leads to necessary decoupling of the fields B_k.

5.2 Canonical quantization on the Light Front

Let us fix the gauge as follows:

$$\partial_- A_- = 0, \qquad A_-^{ij}(x) = \delta^{ij} v^j(x^\perp, x^+). \tag{168}$$

For simplicity below we denote the argument of quantities, not depending on the x^-, again by x. Let us remark that starting with arbitrary field A_μ, periodic in x^-, it is not possible to take zero modes of the A_- equal to zero without a violation of the periodicity. But it is possible to make the A_- diagonal as in the equation (168) [21–24].

Then the action (166) can be written in the form:

$$S = a^2 \sum_{x^\perp} \int dx^+ \int_{-L}^{L} dx^- \Big\{ \sum_i \Big[2F_{+-}^{ii}(x) \partial_+ v^i(x) \Big] + \\ + \frac{1}{(ga)^2} \sum_{i,j} \sum_k \Big[D_- M_k^{ij+}(x) \partial_+ M_k^{ij}(x) + h.c. \Big] + \sum_{i,j} A_+^{ij}(x) Q^{ji}(x) - \mathcal{H}(x) \Big\}, \tag{169}$$

where

$$\begin{aligned} D_- M_k^{ij}(x) &\equiv \Big(\partial_- - igv^i(x) + igv^j(x - e_k) \Big) M_k^{ij}(x), \\ D_- M_k^{ij+}(x) &\equiv \Big(\partial_- + igv^i(x) - igv^j(x - e_k) \Big) M_k^{ij+}(x), \\ D_- F_{+-}^{ij}(x) &\equiv \Big(\partial_- - igv^i(x) + igv^j(x) \Big) F_{+-}^{ij}(x), \end{aligned} \tag{170}$$

the $A_+^{ij}(x)$ play the role of Lagrange multipliers,

$$Q^{ji}(x) \equiv 2D_- F_{+-}^{ji}(x) + \\ + \frac{i}{ga^2} \sum_{j'} \sum_k \Big[M_k^{ij'+}(x) D_- M_k^{jj'}(x) - M_k^{j'j+}(x + e_k) D_- M_k^{j'i}(x + e_k) - \\ - \Big(D_- M_k^{ij'+}(x) \Big) M_k^{jj'}(x) + \Big(D_- M_k^{j'j+}(x + e_k) \Big) M_k^{j'i}(x + e_k) \Big] = 0, \tag{171}$$

are gauge constraints and

$$\mathcal{H} = \sum_{ij} \Big(F_{+-}^{ij\,+} F_{+-}^{ij} + G_{12}^{ij\,+} G_{12}^{ij} \Big) + \mathcal{H}_m \tag{172}$$

is Hamiltonian density. The term $\mathcal{H}_m$ can be obtained from the expression (167) in standard way.

The constraints can be resolved explicitly by expressing the F_{+-}^{ij} in terms of other variables, but zero mode components $F_{+-(0)}^{ii}$ can not be found from constraint equations and play the role of independent canonical variables. Zero modes $Q_{(0)}^{ii}(x^\perp, x^+)$ of the constraints remain unresolved and are imposed as conditions on physical states:

$$Q_{(0)}^{ii}(x^\perp, x^+) \, |\Psi_{phys}\rangle = 0. \tag{173}$$

In order to find complete set of independent canonical variables we write Fourier transformation in x^- of fields $M_k^{ij}(x)$ in the following form:

$$M_k^{ij}(x) = \frac{g}{\sqrt{4L}} \sum_{n=-\infty}^{\infty} \Big\{ \Theta\left(p_n + gv^i(x) - gv^j(x-e_k)\right) M_{nk}^{ij}(x^\perp, x^+) + \\ + \Theta\left(-p_n - gv^i(x) + gv^j(x-e_k)\right) M_{nk}^{ij\,+}(x^\perp, x^+) \Big\} \times \\ \times \left| p_n + gv^i(x) - gv^j(x-e_k) \right|^{-1/2} e^{-ip_n x^-}, \quad (174)$$

where

$$\Theta(p) = \begin{cases} 1, & p > 0 \\ 0, & p < 0 \end{cases}, \qquad p_n = \frac{\pi}{L}\, n, \quad n \epsilon Z. \quad (175)$$

Due to the gauge (168) this Fourier transformation coincides with the expansion in eigen modes of the operator D_-. Therefore the ultraviolet cutoff in these modes, which we will apply, reduces to the following condition on the number of terms in the sum (174):

$$|p_n + gv^i(x) - gv^j(x-e_k)| \le \frac{\pi}{L}\, \bar{n}, \quad (176)$$

where the $\bar{n}$ is integer parameter of ultraviolet cutoff. Let us stress that this regularization is gauge invariant.

The action can be rewritten in the following form (up to nonessential surface terms):

$$S = a^2 \sum_{x^\perp} \int dx^+ \Bigg\{ \sum_i 4L F_{+-(0)}^{ii} \partial_+ v^i + \\ + \frac{i}{a^2} \sum_{i,j} \sum_k {\sum_n}' M_{nk}^{ij\,+} \partial_+ M_{nk}^{ij} + 2L \sum_i A_{+(0)}^{ii} Q_{(0)}^{ii} - \tilde{\mathcal{H}}(x) \Bigg\}, \quad (177)$$

where the $\sum_n'$ means that the sum is cut off by the condition (176), and the $\tilde{\mathcal{H}}$ is obtained from the $\mathcal{H}$ via the substitution of the expression

$$F_{+-}^{ij} = \left(F_{+-}^{ij} - \delta^{ij} F_{+-(0)}^{ii} \right) + \delta^{ij} F_{+-(0)}^{ii}, \quad (178)$$

where the $F_{+-}^{ij} - \delta^{ij} F_{+-(0)}^{ii}$ are to be written in terms of the M_{nk}^{ij}, $M_{nk}^{ij\,+}$, v^i by solving the constraints (171) and using the equation (174). The $F_{+-(0)}^{ii}$ remain independent. The G_{12}^{ij} are also to be expressed in terms of the M_{nk}^{ij}, $M_{nk}^{ij\,+}$, v^i via the equations (164), (174).

We have the following set of canonically conjugated pairs of independent variables:

$$\left\{ v^i, \quad \Pi^i = 4La^2 F_{+-(0)}^{ii} \right\}, \qquad \left\{ M_{nk}^{ij}, \quad i M_{nk}^{ij\,+} \right\}. \quad (179)$$

In quantum theory these variables become operators which satisfy usual canonical commutation relations:

$$[v^i(x), \Pi^j(x')]_{x^+=0} = i\delta^{ij} \delta_{x^\perp, x'^\perp}, \\ [M_{nk}^{ij}(x), M_{n'k'}^{i'j'\,+}(x')]_{x^+=0} = \delta^{ii'} \delta^{jj'} \delta_{nn'} \delta_{kk'} \delta_{x^\perp, x'^\perp}; \quad (180)$$

the other commutators being equal to zero.

Let us remark that the condition (168) does not fix the gauge completely. In particular, discrete group of gauge transformations, depending on the x^-, of the form

$$U_n^{ij}(x) = \delta^{ij} \exp\left\{ i\frac{\pi}{L} n^i(x^\perp) x^- \right\}, \tag{181}$$

where $n^i(x^\perp)$ are integers, remains, and, of course, transformations, not depending on the x^-, are allowed. Under the transformations (181) canonical variables change as follows:

$$\begin{aligned} v^i(x) &\longrightarrow v^i(x) - \frac{\pi}{gL}\, n^i(x^\perp), \qquad \Pi^i \longrightarrow \Pi^i, \\ M_{nk}^{ij}(x^\perp) &\longrightarrow M_{n'k}^{ij}(x^\perp), \quad n' = n + n^i(x^\perp) - n^j(x^\perp - e_k). \end{aligned} \tag{182}$$

Let us write the expression for quantum operators $Q_{(0)}^{ii}(x^\perp, x^+)$, which define the physical subspace of states. We fix the order of the operators in such a way as to relate with classical expression $G_{\mu\nu}^+ G^{\mu\nu}$ a quantum one of the form:

$$\frac{1}{2}\left(G_{\mu\nu}^+ G^{\mu\nu} + G^{\mu\nu} G_{\mu\nu}^+ \right). \tag{183}$$

We remark that other choices of the ordering do not admit reasonable vacuum solution. Then the operators $Q_{(0)}^{ii}(x^\perp, x^+)$ have the following form in terms of canonical variables:

$$\begin{aligned} 2LQ_{(0)}^{ii}(x^\perp, x^+) = -\frac{g}{4a^2} \sum_j \sum_k {\sum_n}' \Big[\varepsilon\left(p_n + gv^j(x+e_k) - gv^i(x) \right) \times \\ \times \left({M_{nk}^{ji}}^+(x+e_k) M_{nk}^{ji}(x+e_k) + M_{nk}^{ji}(x+e_k) {M_{nk}^{ji}}^+(x+e_k) \right) - \\ -\varepsilon\left(p_n + gv^i(x) - gv^j(x-e_k) \right) \left({M_{nk}^{ij}}^+(x) M_{nk}^{ij}(x) + M_{nk}^{ij}(x) {M_{nk}^{ij}}^+(x) \right) \Big], \end{aligned} \tag{184}$$

where

$$\varepsilon(p) = \begin{cases} 1, & p > 0 \\ \text{-1}, & p < 0 \end{cases}. \tag{185}$$

One can easily construct canonical operator of translations in the x^-:

$$\begin{aligned} P_-^{can} = \frac{1}{2} \sum_{x^\perp} \sum_{i,j} \sum_k {\sum_n}' p_n\, \varepsilon\left(p_n + gv^i(x) - gv^j(x-e_k) \right) \times \\ \times \left({M_{nk}^{ij}}^+(x) M_{nk}^{ij}(x) + M_{nk}^{ij}(x) {M_{nk}^{ij}}^+(x) \right). \end{aligned} \tag{186}$$

This expression differs from the physical gauge invariant momentum operator P_- by a term proportional to the constraint. The operator P_- is

$$\begin{aligned} P_- = \frac{a^2}{2} \sum_{x^\perp} \sum_k \int_{-L}^{L} dx^- \, \mathrm{Tr}\left(G_{-k}^+ G_{-k} + G_{-k} G_{-k}^+ \right) = P_-^{can.} + 4La^2 \sum_{x^\perp} \sum_i v^i Q_{(0)}^{ii} = \\ = \frac{1}{2} \sum_{x^\perp} \sum_{i,j} \sum_k {\sum_n}' \left| p_n + gv^i(x) - gv^j(x-e_k) \right| \times \\ \times \left({M_{nk}^{ij}}^+(x) M_{nk}^{ij}(x) + M_{nk}^{ij}(x) {M_{nk}^{ij}}^+(x) \right) = \\ = \sum_{x^\perp} \sum_{i,j} \sum_k {\sum_n}' \left| p_n + gv^i(x) - gv^j(x-e_k) \right| \left({M_{nk}^{ij}}^+(x) M_{nk}^{ij}(x) + \frac{1}{2} \right). \end{aligned} \tag{187}$$

Let us choose a representation of the state space, in which the variables $v^i(x)$ are the multiplication operators. The states are described in this representation by normalizable functionals $F[v]$ of classical functions $v^i(x)$ (in fact by functions, depending on the values of the v^i in different points $x^\perp$ due to the discreteness of these $x^\perp$). One can define full space of states as direct product of Fock space, in which the ${M_{nk}^{ij}}^+(x)$ and $M_{nk}^{ij}(x)$ play the role of creation and annihilation operators, and the space of functionals $F[v]$. Let us call M-vacuum the states of the form $|0\rangle \cdot F[v]$, where the $|0\rangle$ satisfies the condition

$$M_{nk}^{ij}(x)|0\rangle = 0, \tag{188}$$

and the F is some functional. Arbitrary state can be represented in the form of linear combination of vectors $|\{m\}; F\rangle$ of the form

$$\prod_{x^\perp}\prod_{i,j}\prod_{k}\prod_{n}{}' \left({M_{nk}^{ij}}^+(x)\right)^{m_{nk}^{ij}(x)} |0\rangle \cdot F[v] \tag{189}$$

with different nonnegative integer functions $m_{nk}^{ij}(x^\perp)$ and functionals F. One can define orthonormalized set of such functionals if necessary.

One can see from (187) that the state, corresponding to the absolute minimum of the P_- must satisfy the conditions (188), i.e. to be a M-vacuum. The value of the P_- in this state can be written in the form

$$\langle 0; F|\, P_- \,|\, 0; F\,\rangle = \\ = \frac{1}{2}\int \prod_{x^\perp}\prod_{i} dv^i(x) \sum_{x^\perp}\sum_{i,j}\sum_{k}\sum_{n}{}' \left|p_n + gv^i(x) - gv^j(x-e_k)\right| |F[v]|^2. \tag{190}$$

Remind that the $\sum\limits_{n}{}'$ denotes the sum in n limited by the condition (176). If one uses this condition and shifts the index n in these sums by integer part of the quantities $(gL(v^i(x) - v^j(x-e_k))/\pi)$, one sees that the dependence on the v^i cancels in sums over n and that the expression (190) does not depend on the $F[v]$ if it is normalized to unity. Thus the momentum P_- has the minimum in all M-vacua. One can make the value of the P_- in these vacua equal to zero by subtracting corresponding constant from the operator P_-.

Let us show that M-vacua are the physical states, i. e. satisfy the condition (173). Indeed, in M-vacua this condition looks as follows:

$$\sum_{j}\sum_{k}\sum_{n}{}' \left[\varepsilon\left(p_n + gv^j(x+e_k) - gv^i(x)\right) - \varepsilon\left(p_n + gv^i(x) - gv^j(x-e_k)\right)\right] F[v] = 0 \tag{191}$$

and is satisfied for any $F[v]$, because for every link in the sum (191) the numbers of positive and negative values of the ε-functions are equal. For arbitrary basic states (189) analogous conditions have the following form:

$$\sum_{j}\sum_{k}\sum_{n}{}' \left[\varepsilon\left(p_n + gv^j(x+e_k) - gv^i(x)\right) m_{nk}^{ij}(x+e_k) - \\ - \varepsilon\left(p_n + gv^i(x) - gv^j(x-e_k)\right) m_{nk}^{ij}(x)\right] F[v] = 0. \tag{192}$$

The eigen values p_- of the operator P_- can be found from the equation

$$\sum_{x^\perp}\sum_{i,j}\sum_{k}\sum_{n}{}' \left|p_n + gv^i(x) - gv^j(x-e_k)\right| m_{nk}^{ij}(x) F[v] = p_- F[v] \tag{193}$$

where the functional $F[v]$ is normalized.

To define physical vacuum state correctly one must consider not only states, corresponding to the minimum of the operator P_-, but also to the minimum of the operator P_+. One can try to do this via minimization of the P_+ on the M-vacua, i. e. on the set of states with $p_- = 0$. The expression $\langle 0|P_+|0\rangle$, where $|0\rangle$ is the Fock space vacuum w. r. t. the M_{nk}, ${M_{nk}}^+$, depends on the functions $v^i(x)$ (which enter, in particular, into the "normal contractions" of the operators M_k, M_k^+) and on the operators Π^i, canonically conjugated to the $v^i(x)$. The expectation value of this expression is to be minimized on the set of functionals $F[v]$. The resulting functional $F[v]$ must decrease in the vicinity of those values of the $v^i(x)$, for which the operator D_- has zero eigen value, because the Hamiltonian is singular at these values (it is seen, for example, from the expansion (174)).

The vacuum state constructed in such a way strongly differs from the usual vacuum in continuous space theory, because for M-vacuum we get zero expectation values of the operators M_k, but not of the operators $(M_k - I)/ga = B_k + iA_k$, related to usual gauge fields. Beside of this the condition of the unitarity of the matrices M_k in the continuum limit (or equivalent condition of switching off the nonphysical fields B_k) cannot be fulfilled in such a vacuum. This disagreement with conventional theory is caused by the absence of explicit Lorentz invariance in our formulation, that leads to different quantum states, corresponding to the minima of the operators P_- and P_+. It is not clear whether these states can be made coinciding at least in the limit of continuous space. This requires further investigation.

Nevertheless the method of the quantization of gauge theories on the LF, described here, can be useful for completely gauge-invariant formulation of some effective models, based on analogous formalism (but without complete gauge invariance due to throwing out of all zero modes of fields and due to the absence of gauge-invariant regularization of ultraviolet divergencies in the p_-). Such models are described, for example, in papers [45, 46], where the ideas of papers [47, 48] were developed.

6 Conclusion

The LF Hamiltonian approach to Quantum Field Theory briefly reviewed here is an attempt to apply a beautiful idea of Fock space representation for quantum field nonperturbatively in the framework of canonical formulation on the LF. The problem of describing the physical vacuum state becomes formally trivial in this approach because such vacuum state coincides with mathematical vacuum of LF Fock space.

However the breakdown of Lorentz and gauge symmetries due to regularizations generates difficulties in proving the equivalence of LF formalism and the usual one in Lorentz coordinates. This problem can be solved for nongauge theories but turns out to be very difficult for gauge theories in special (LF) gauge which is needed here. Nevertheless we hope that these difficulties can be overcome by finding a modified form of canonical LF Hamiltonian which generates the perturbation theory equivalent to usual covariant and gauge invariant one.

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Appendix 1

Statement 1. *If conditions (112) are satisfied, then, for fixed external momenta p^s and $p_-^s \neq 0\ \forall s$, the equality*

$$\lim_{\beta\to 0}\lim_{\gamma\to 0}\int\prod_k dq_+^k \int_{V_\varepsilon}\prod_k dq_-^k \frac{\tilde f(Q^i,p^s)e^{-\gamma\sum_i {Q_+^i}^2-\beta\sum_i {Q_-^i}^2}}{\prod_i(2Q_+^iQ_-^i-M_i^2+i\text{æ})} =$$

$$= \int\prod_k dq_+^k \int_{V_\varepsilon\cap B_L}\prod_k dq_-^k \frac{\tilde f(Q^i,p^s)}{\prod_i(2Q_+^iQ_-^i-M_i^2+i\text{æ})}, \qquad (A.1.1)$$

holds while the expressions appearing in (A.1.1) exist and the integral over $\{q_+^k\}$ on the right-hand side is absolutely convergent. It is assumed that the momenta of lines Q^i are expressed in terms of loop momenta q^k, V_ε is the domain corresponding to the presence of full lines, type-1 lines, and type-2 lines (the definitions are given following formula (123)), B_L is the sphere of radius L, where $L \geq S \max_s |p_-^s|$, and S is a number depending on the diagram structure.

Let us prove the statement. For each type-1 line in (A.1.1), we perform the following partitioning:

$$\int_{-\varepsilon}^{\varepsilon} dQ_-^i = \left[\int dQ_-^i + (-1)\left(\int_{-\infty}^{-\varepsilon} dQ_-^i + \int_{\varepsilon}^{\infty} dQ_-^i\right)\right]. \qquad (A.1.2)$$

Then both sides of equation (A.1.1) become the sum of expressions of the same form in which, however, the domain V_ε corresponds to the presence of only full and type-2 lines. It is clear that by proving the statement for this V_ε: (which is done below), we prove the original statement as well.

Let $\tilde B$ be a domain such that the surfaces on which $Q_-^i = 0$ are not tangent to the boundary $\tilde B$. First, we prove that in the expression

$$\int\prod_k dq_+^k \int_{V_\varepsilon\cap\tilde B}\prod_k dq_-^k \frac{\tilde f(Q^i,p^s)\, e^{-\beta\sum_i {Q_-^i}^2}}{\prod_i(2Q_+^iQ_-^i-M_i^2+i\text{æ})} \qquad (A.1.3)$$

the integral over $\{q_+^k\}$ is absolutely convergent (here the integral over $\{q_-^k\}$ is finite because xæ > 0, $\beta > 0$). This becomes obvious (considering conditions (112) and the fact that, in type-2 lines, the momentum Q_-^i is separated from zero) if the contours of the integration over $\{q_-^k\}$ can be deformed in such a way that the momenta Q_-^i of the full lines are separated from zero by a finite quantity (within the domain $V_\varepsilon \cap \tilde B$). In this case, we can repeat the well-known Weinberg reasoning [51]. What can prevent deformation is either a "clamping" of the contour or the point $Q_-^i = 0$ falling on the integration boundary.

Let us investigate the first alternative. We divide the domain of integration over q_+^k into sectors such that the momenta of all full lines Q_+^i have a constant sign within one sector. Let us examine one sector. We take a set of full lines whose Q_-^i may simultaneously vanish. In the vicinity of the point where Q_-^i from this set vanish simultaneously, we bend the contours of the integration over $\{q_-^k\}$ such that these contours pass through the points $Q_-^i = iB^i$ and the momenta Q_-^i of the type-2 lines do not change. Let B^i be such that $B^iQ_+^i \geq 0$ for the lines from the set (for Q_+^i from the sector under consideration). It is easy to check that this bending

is possible. (Since the contours of integration over q_-^k are bent and Q_-^i are expressed in terms of q_-^k, one should only check that such b^k exist, where the necessary B^i are expressed in the same way, i.e., that B^i obey the conservation laws and flow only along the full lines). With this bending, rather small in relation to the deviation and the size of the deviation region, the contours do not pass through the poles because, for the denominator of each line from the set in question, we have

$$\left(2Q_+^iQ_-^i - M_i^2 + i\text{æ}\right) \to \left(2Q_+^i\left(Q_-^i + iB^i\right) - M_i^2 + i\text{æ}\right), \quad Q_+^iB^i \geq 0, \tag{A.1.4}$$

and for the other denominators, the bending takes place in a region separated from the point where the corresponding momenta Q_-^i are equal to 0. Repeating the reasoning for all sets, we can see that there is no contour "clamping".

The other alternative is excluded by the above condition for $\tilde{B}$. To make this clear, one should introduce such coordinates ξ^α in the q^k-space that the boundary of the domain $\tilde{B}$ is determined by the equation $\xi^1 = a = const$ and then argue as above for the coordinates ξ^α with $\alpha \geq 2$.

After bending the contours, integral (A.1.3) is absolutely convergent in q_+^k, q_-^k if the integration in q_+^k is carried out within the sector under consideration. On pointing out that the result, of internal integration in (A.1.3) does not depend on the bending, we add the integrals over all sectors and conclude that (A.1.3) converges in $\{q_+^k\}$ absolutely.

Now let us prove that if $\tilde{B}$ is a quite small, finite vicinity of the point $\{\tilde{q}_-^k\}$ that lies outside the sphere B_L, then expression (A.1.3) is equal to zero. We consider the momentum Q_-^i of one line. Flowing along the diagram, it can ramify or it can merge with other momenta. Clearly, two situations are possible: either it flows away completely through external lines, or, probably, after long wandering, part of it, $\tilde{Q}_-$, makes a complete loop. The former situation is possible only if $|Q_-^i| \leq \sum_r |p_-^r|$, where all external momenta leaving the diagram (but not entering it) are summed. Obviously, S can be chosen such that for $\{q_-^k\}$ from $\tilde{B}$, a line exists whose momentum violates this condition.

The latter situation results in the existence of a loop, where the inequality $Q_-^i > \tilde{Q}_-$ holds for all momenta of its lines and the positive direction of the momenta is along the loop. Then the integral over q_+^k of the loop in question can be interchanged with the integrals over $\{q_-^k\}$ (because it is absolutely and uniformly convergent for all q_-^k) and the residue formula can be used to perform this integration. Since, for the loop in question, the momenta Q_-^i of the lines of this loop are separated from zero and are of the same sign, the result is zero. This has a simple physical meaning. If we pass to stationary noncovariant perturbation theory, we find that only quanta with positive Q_- can exist. In this case, external particles with positive p_- are incoming and those with negative p_- are outgoing. Then, the momentum conservation law favors the occurrence of the first situation.

The entire outside space for $\tilde{B}$ can be composed of the above domains B_L (everything converges well at infinity due to the factor $\exp(-\beta \sum_i {Q_-^i}^2)$). Thus, on the left-hand side of (A.1.1), one can substitute the integration domain $V_\varepsilon \cap B_L$ for V_ε, set the limit in γ under the sign of integration over $\{q_+^k\}$ because of its absolute convergence, and also set the limit in β under the integration sign because the domain of the integration over $\{q_-^k\}$ is bounded. Thus, we obtain the right-hand side. The statement is proved.

Statement 2. *If V_ε corresponds to the presence of type-2 lines alone, then, under the same*

conditions as in Statement 1, the equality

$$\int\limits_{V_\varepsilon} \prod_k dq_-^k \int \prod_k dq_+^k \frac{\tilde{f}(Q^i, p^s)}{\prod_i (2Q_+^i Q_-^i - M_i^2 + i\text{æ})} =$$

$$= \int \prod_k dq_+^k \int\limits_{V_\varepsilon \cap B_L} \prod_k dq_-^k \frac{\tilde{f}(Q^i, p^s)}{\prod_i (2Q_+^i Q_-^i - M_i^2 + i\text{æ})}. \quad (A.1.5)$$

is valid.

The proof of this statement is analogous to the second part of the proof of Statement 1.

Appendix 2

Statement. *If conditions (112) are satisfied, the limits in γ and β in (118) can be interchanged (in turn) with the sign of the integral over $\{\alpha_i\}$ and then with $\tilde{f}\left(-i\frac{\partial}{\partial y_i}\right)$.*

To prove this, we define the vectors

$$\{q_+^1, q_-^1, \ldots, q_+^l, q_-^l\} \equiv S, \qquad \{Q_+^1, Q_-^1, \ldots, Q_+^n, Q_-^n\} \equiv \mu S + P,$$
$$\{y_1^+, y_1^-, \ldots, y_n^+, y_n^-\} \equiv Y, \quad (A.2.1)$$

where the vector P is built only from external momenta and μ is an $l \times n$ matrix of rank l, $\mu_{2k-1}^{2i} = \mu_{2k}^{2i-1} = 0$, $\mu_{2k}^{2i} = \mu_{2k-1}^{2i-1}$. Next, we introduce the following notation:

$$\tilde{\Lambda}_i = \begin{pmatrix} \gamma & -i\alpha_i \\ -i\alpha_i & \beta \end{pmatrix}, \quad \Lambda = diag\{\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_n\}, \quad A = \mu^t \Lambda \mu,$$
$$B = \mu^t \Lambda P - \frac{1}{2} i \mu^t Y, \quad C = -P^t \Lambda P + i Y^t P - i \sum_i \alpha_i M_i^2. \quad (A.2.2)$$

Then it follows from (119) that

$$\hat{\varphi}(\alpha_i, p^s, \gamma, \beta) = (-i)^n \tilde{f}\left(-i\frac{\partial}{\partial y_i}\right) \int d^{2l}S\, e^{-S^t A S - 2B^t S + C}\Big|_{y_i=0} =$$

$$= (-i)^n \tilde{f}\left(-i\frac{\partial}{\partial y_i}\right) e^{B^t A^{-1} B + C} \frac{\pi^l}{\sqrt{\det A}}\Big|_{y_i=0}. \quad (A.2.3)$$

The function $\tilde{f}$ is a polynomial and we consider each of its terms separately. Up to a factor, each term has the form $\frac{\partial}{\partial y_{i_1}} \ldots \frac{\partial}{\partial y_{i_r}}$. These derivatives act on C and B. The action on C results in the constant factor $iN^t P$, the action on B results in the factor $-(1/2)iN^t \mu A^{-1} B$ or $-(1/4)N_1^t \mu A^{-1} \mu^t N_2$ (the latter is the result of the action of two derivatives; N, N_1, and N_2 are constant vectors).

It is necessary to prove the correctness of the following three procedures: (i) setting the limit in γ under the integral sign for fixed $\beta > 0$; (ii) setting the limit in β for $\gamma = 0$; (iii) setting the limits in γ and β under the signs of differentiation with respect to Y. In cases (i) and (ii), one must obtain the bounds

$$|\hat{\varphi}(\alpha_i, p^s, \gamma, \beta)| \leq \varphi'(\alpha_i, p^s, \beta), \quad (A.2.4)$$

$$|\hat{\varphi}(\alpha_i, p^s, 0, \beta)| \leq \varphi''(\alpha_i, p^s), \tag{A.2.5}$$

where φ' and φ'' are functions integrable (for φ' if $\beta > 0$) in any finite domain over α_i, with $\alpha_i \geq 0$. Then, for case (i), we have

$$|\hat{\varphi}(\alpha_i, p^s, \gamma, \beta)\, e^{-æ\sum_i \alpha_i}| \leq \varphi'(\alpha_i, p^s, \beta)\, e^{-æ\sum_i \alpha_i}, \tag{A.2.6}$$

i.e., a limit on the integrated function arises, and, thus, the limit in γ can be put under the integral sign. The situation is similar for case (ii). It is evident from (A.2.3) that the function $\hat{\varphi}$ can be singular only if the eigenvalues of matrix A become zero. On finding the lower bound of these eigenvalues, one can prove through rather long reasoning that bounds (A.2.4), (A.2.5) exist if condition (112) is satisfied.

After the limits in γ and β are put under the integral sign, it is not difficult to interchange them with the differentiation with respect to Y. One need do it only for $\alpha_i > 0$ (for each i) and, in this case, one can show that the eigenvalues of the matrix A are nonzero and $\hat{\varphi}$ is not singular.

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